THE CAPTURE OF MESONS

by

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STATEMENT

The work described in this thesis was undertaken in order to provide a theory of the capture of pi mesons in light elements , particularly in helium and carbon. The nature of the pi mesons was only imperfectly understood , and while they were widely thought to be connected in some way with Yukawa's theory of nuclear forces , the nature of the connection could only be guessed at. Assuming the pi mesons to be of a type considered by Kemmer , there are still four possible ways of describing them , of which the scalar and pseudoscalar theories seem to hold most promise. It was hoped that a simple theory of meson capture , giving the enrgy spectra of the emitted particles and the relative probabilities of different reactions , would lead to information about the kind of theory required for the description of the pi mesons.

It is shown that no information about the meson interaction would be gained from the kind of experiments proposed. However , some interesting results emerge from the theory of meson capture by the carbon nucleus , which is treated by means of a simple alpha - particle model. The energy spectrum and angular correlation function of the emitted alpha particles are calculated. Comparison with experiment leads to information about the nuclear wave functions and also shows that some of the disintegrations involve the emission of beryllium - 8 as an intermediate state.

Most of this work was undertaken in collaboration with Miss S. N. Ruddlesden , but Section 3. 5 on the angular distributions of the neutrons emitted from helium-4 , Section 5. 5 on the indirect disintegration of carbon-12 and Section 5. 6 on the angular correlations of the alpha particles from carbon-12 , are entirely due to the author. In addition , the considerations on angular momentum and parity involving the use of the wave functions defined in Section 3. 1 et seq., and the calculations with wave functions of "asymptotic" type defined in Section 2. 2 are entirely due to the author. The numerical integrations required for the calculation of the alpha-particle energy spectrum in Section 5. 3 are entirely due to Miss Ruddlesden.

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INTRODUCTION

1. 1. Historical.

The foundations of the modern theory of nuclear forces were laid by Yukawa (1935). Guided by the theory of interaction of electrically charged particles through the electromagnetic field, Yukawa postulated the existence of a new kind of field, later called the meson field, capable of interacting with nuclear particles. The field equations adopted by Yukawa, when quantized by the well-known rules of quantum mechanics, lead to the existence of charged particles of a certain rest mass and obeying Bose-Einstein statistics. This simple assumption had far-reaching effects, for by choosing the mass of the field-quanta or mesons to be about 100 electron masses and with a suitable choice of coupling constant, the force between proton and neutron could be explained in a reasonable manner. By further introducing an interaction between the meson field and the electron-neutrino field of Fermi, Yukawa explained in a qualitative way the radioactive β-disintegration of nuclei. He further predicted that the meson would be unstable against β-decay with a lifetime of about 10^{-6} secs.

The discovery by Neddermeyer and Anderson (1937) of charged particles in the cosmic radiation with about 200 times the mass of the electron was regarded as confirmation of Yukawa's theory, but already certain theoretical difficulties had become apparent which prevented an exact correspondence between experimental and theoretical results.

The failure of Yukawa's theory to give the right proton-neutron force in the singlet state of the deuteron led many authors ,notably Kemmer (1938a) to attempt its generalization. This resulted in the four kinds of meson theory , namely,scalar, pseudoscalar,vector and pseudovector, which are characterized by the spin and parity of the meson. Mixtures of different kinds of mesons were considered by Schwinger (1942) and by Møller and Rosenfeld (1940) who tried to separate out the static part of the nuclear forces, By this time it was generally felt that the meson theory was satisfactory as a qualitative explanation of the nuclear forces, but that the enormous complexity of the equations prevented an accurate calculation of the quantities to be compared with experiment.

The interactions between mesons and nucleons had been studied experimentally by examining the fate of mesons stopped in matter. A theoretical discussion by Tomonaga and Araki (1940) led to the conclusion that the time for the fast meson to be slowed down by electronic collisions was so small that most mesons would not be captured in flight by a nucleus. After being slowed down, a positive meson would be repelled by the nuclear charge and would decay whereas a negative meson would be attracted into a closed orbit around a nucleus and would initiate a nuclear disintegration. This view was upheld by the experiments of Conversi and Piccioni (1946), who showed that about half the mesons stopped in solids gave rise to decay electrons.

The most refined experiments on these lines were made by Conversi, Pancini and Piccioni (1947), who studied positive and negative mesons separately. They showed that when mesons were stopped in iron the early results were reproduced and only the positive mesons decayed, but both positive and negative mesons decayed when stopped in carbon. Calculations made by Fermi and Teller (1947) showed that a negative meson of 2keV energy will be captured into the lowest Bohr orbit about a nucleus in about 10^{-13} secs, i.e. in a time much less than the lifetime against decay. The time required for the meson to be captured into the nucleus, causing disintegration, was calculated to be about 10^{-18} secs on the conventional meson theory of nuclear forces.

As a possible solution of this problem, Fröhlich (1947) suggested that the meson might be trapped by falling into a state with a very long lifetime, so that it decayed before getting into one of the characteristic Bohr orbits. This suggestion was followed up by Rosenberg (1949) and Huby (1949), who showed that the existence of traps was in some circumstances possible, though not very likely. These researches also provided confirmation of the calculations of Fermi and Teller of the slowing-down time, which, being based on the free-electron model of solids, were not very accurate.

At this time it was shown that a charged particle lighter than a proton could be detected in a photographic emulsion, which discovery led to the highly developed technique of studying nuclear particles at present in use. Using photographic plates exposed at high altitude, the Bristol group showed that there are at least two kinds of mesons, denoted by π and μ. They showed that π-mesons are emitted in nuclear collisions in the upper atmosphere, but their short life (later found to be 10^{-8} secs) prevents them reaching sea-level. The μ-mesons are always observed as decay products of π-mesons, and are the only ones found at sea-level, Their interactions with nuclei are supposed to be very small, in agreement with the experiments mentioned above. The production of mesons by the Berkeley cyclotron and subsequent experiments has confirmed all these opinions. It is possible that there is no direct interaction between μ-mesons and nucleons, and that nuclear reactions initiated by μ-mesons proceed via the mutual coupling with π-mesons. These ideas had been discussed by Marshak and Bethe (1947) even before the discovery of the π-meson, though their proposed scheme differs in detail from the one now generally accepted. Because of their strong interaction, it is possible to regard the π-mesons as the ones described by Yukawa's theory, although the recent discovery of the τ-meson (Rochester and Butler 1947), which has a strong nuclear interaction, has introduced fresh complexity into the study of nuclear forces.

One of the most interestihg features of the π-mesons, and one of the means by which they are identified in practice, is their ability to produce stars in the emulsion. This effect is interpreted as the capture of a π-meson by a nucleus which subsequently disintegrates. Owing to the composition of the emulsion, most of the stars arise from silver or bromine nuclei and have one or two 'prongs', but occasionally one of the light nuclei, carbon, nitrogen or oxygen, present in the gelatin, captures a meson, the resulting star often having three or more prongs. The occurrence of stars in light nuclei was conclusively proved by Menon, Muirhead and Rochat (1950)), referred to as M.M.R., who used'sandwiches' consisting of a thin layer og gelatin between layers of photographic emulsion, so that charged particles emitted from the nuclei in the gelatin would leave a track in the latter part of their range.

The recent discovery of heavy mesons τ and κ, both charged and neutral, makes the simple picture outlined above much more complicated and has led some authors to attempt to formulate a unified theory in which mesons can have different allowed values of mass and spin, as yet without marked success. It appears that the general picture of meson and nucleon fields in interaction is substantially correct but that a detailed theory is impossible until ways are found of handling the complicated calculations.

1. 2 The Interactions of Mesons

The experiments on the stopping of slow π^--mesons in photographic emulsions show them to have a strong interaction with nucleons. The relative yields from the Berkeley cyclotron of positive and negative π-mesons shows symmetry between them, and the existence of a neutral π-meson, π^0, (Bjorkland et. al. 1950) indicates that a theory on the lines of Kemmer's (1938b) symmetrical theory may have some success. The π - μ decay and the τ - π decay show that we have a very complicated scheme of several kinds of meson field, nucleons, electrons and neutrinos, all in mutual interaction. The forces between low energy nucleons should, however, only depend on the π-mesons, since the μ-mesons only interact weakly with nucleons. The greater mass of the τ and K mesons should reduce their efficacy as agents of nuclear forces.

It is consistent to assume that the π-mesons have integral spin and that the μ-mesons have spin $\frac{1}{2}$. The capture of a μ^- by a nucleus must thus be accompanied by the emission of a neutrino in order to conserve angular momentum. This explains why the time for interaction is so long and why the nuclear particles receive so little of the meson's rest energy. For the π^--meson interaction we consider the simplest capture processes, as follows:

$$P + \pi^- \longrightarrow N \qquad (1)$$

$$P + \pi^- \longrightarrow N + \pi^0 \qquad (2)$$

$$P + \pi^- \longrightarrow N + \gamma \qquad (3)$$

where P, N, γ mean proton, neutron and photon respectively. Conservation of momentum requires that (1) can only take place in a complex nucleus, but (2) and (3) can occur when P is a free proton. We can see immediately that process (1), if it can occur, will be much more probable than (2) or (3).

According to the best values of the masses of π^- and π^0 mesons, 275 and 265 times the electron mass respectively, the Q-value of reaction (2) is about 4MeV; this is only just sufficient to overcome the binding energy of the deuteron, and therefore (2) can effectively only occur with free protons. The probability of (3) will be small because it depends on the weak coupling between charged particles and Maxwell field, due to the smallness of the fine-structure constant.

The capture of π^- mesons by free protons has been studied by Aamodt et. al. (1950), who found that the spectrum of the emitted γ-rays consisted of two parts: a sharp line at 130 MeV which could only be due to reaction (3) and a square pulse centred at 70 MeV. This latter is interpreted

interpreted as being due to reaction (2) , the π^0 meson decaying into two γ-rays. Because of the small kinetic energy of the π^0 , the two γ-rays have nearly the same energy. Measurement of the width of the pulse , interpreted as Doppler broadening , enabled the mass of the π^0 to be calculated with high accuracy. The fact that it is so close to the mass of the π^- meson justifies us in calling these mesons by the same name , the small mass difference perhaps being accountable as an electromagnetic effect. The selection rules governing the decay of a neutral meson into two γ-rays were investigated by Yang (1950) , who found that this mode of decay is forbidden to mesons of spin 1. He further showed that the planes of polarization of the γ-rays are parallel or perpendicular according as the meson is scalar or pseudoscalar , but no experimental check on this point seems to have been made.

Rather surprizingly , the capture of π^- mesons by deuterons depends more on the character of the meson than in tha case of capture by protons. This is because selection rules of angular momentum and parity are very importantant, due to the presence in the final state of two neutrons , which must obey the Pauli exclusion principle. The γ-rays from this reaction were examined by Aamodt et. al., who found a line at 130 MeV but no continuous spectrum. This means that reactions (1) and (3) are competing and (2) is absent. Process (1) was found to be about twice as probable as process (3). The fact that rection (1) can occur allows us to deduce that the π^- cannot be scalar. According to a calculation by Tamor (1951) the vector π^- meson would give a very small probability for reaction (3) ; thus this case can probably be excluded also. The weight of evidence from these experiments seems to indicate that both charged and neutral π -mesons are pseudoscalar , although the possibility must be retained that the π^0 is scalar or the $\pi^\pm$ is pseudovector.

1. 3 Nuclear Disintegrations Produced by π -mesons

The experiments of the Bristol group showed that stars produced by π^- mesons fall into two groups , according to the number of prongs , i.e. tracks left by charged particles emitted in the disintegration. The stars with one or two prongs were mostly assigned to silver and bromine nuclei. It was shown that meson capture by carbon , nitrogen and oxygen nuclei often gave stars with three or four prongs , some of which were nearly always made by α -particles. The disintegrations of light nuclei , with particular reference to α -particle emission from C^{12} , is the main subject of the present work , but it is useful to consider the disintegration of a heavy nucleus in order to bring out the points of difference.

The ideas required for an understanding of reactions in heavy nuclei are due to Bohr (1936), who pointed out that if a nucleus composed of strongly interacting particles is excited by an external stimulus such as a bombarding particle, the energy of excitation will rapidly be dispersed throughout the nucleus. Only after a comparatively long time will enough energy be concentrated in one particle to enable it to escape. Thus it is possible to define a temparature and to treat the excited nucleus by the methods of statistical thermodynamics. This was done by Weisskopf (1937), who showed that the emission of particles from an excited nucleus was analogous to the evaporation of molecules from a liquid drop. The method of Weisskopf was improved by Le Couteur (1950), who took account of the competition between various emission processes which obtain at high excitation energies. Le Couteur's predictions for the stars from silver and bromine produced by fast neutrons and slow π^- mesons agree very well with the experimental results.

The successful application of evaporation theory requires that the excitation energy be less than the binding energy. In the case of π^- meson capture by C^{12} the excitation is about 140 MeV and the binding energy about 90 MeV. The situation is even worse in the case of

α-particle emission ; the binding energy against disintegration into two α-particles and four nucleons being about 36 MeV. In addition one requires the number of particles in the nucleus to be large enough for approximate thermodynamic equilibrium to be attained before emission of the first particle. It is clear that these conditions are not satisfied for nuclei lighter than say O^{16} . Nevertheless it is possible that evaporation theory may have some application even in this case , for example if one of the protons in the original nucleus is converted into a fast neutron which is quickly emitted , leaving a residual neucleus without much excitation energy which subsequently evaporates one or two particles.

If the capturing nucleus is very light , say He^4, one cannot imagine a compound nucleus , and the disintegration must be pictured as a single event. In this case all the rest energy of the meson will be shared between the four nucleons in a roughly symmetrical way , the only other limitation on the behaviour of the nucleons being conservation of momentum. Since the disintegration is a direct process , the meson interaction may have an effect on such observable quantities as (a) relative probabilities of different reactions and (b) energy spectra of emitted particles. We have already seen that this is true of meson reactions in H^1 and H^2 and that angular momentum and parity requirements very severely restrict the number of permitted final states. In the case of meson captuee by He^4, the number of final states with proton emission is very large , but even so the energy spectrum of the emitted protons may depend enough on the meson interaction for some conclusions about the latter to be drawn.

He^4 is the lightest nucleus , after H^1 and H^2, on which meson-capture experiments may reasonably be expected , since H^3 and He^3 are very rare , and also the disintegration of He^4 leads to a singly-charged particle and neutrons. The theory of slow meson reactions in He^4 is discussed in §§ 2 and 3 , where the relative probabilities of the different processes , energy spectra and angular correlation functions of the emitted particles are calculated.

The great stability of He^4 has led many authors to consider a model of a heavy nucleus , which is regarded as built up of α-particles. This theory reached its highest point in the hands of Wheeler (1937) , who pictured the α-particles , deuterons and possibly other sub-units as constantly dissolving and reforming.

In the region of low atomic weight ,the "α-particle nuclei" C^{12} and O^{16} are known to be more stable than the adjacent species. After meson capture , these nuclei should emit α-particles relatively often , as found experimentally by M.M.R. This in itself is not evidence in favour of the α-particle model , since the high binding energy of the α- particle gives much bigger Q values for α-particle emission. However, the α-particle model of light nuclei has the advantage of simplicity , and in § 5 the capture of slow mesons by C^{12} with emission of two α-particles is considered and the energy spectra and angular correlation of the α-particles are compared with experiment.

2. π^--meson Capture by He^4

2. 1. General Considerations

The theoretical work previously referred to on the slowing down of negative π-mesons shows that in all substances the meson comes to fest before decaying. The slow meson will lose energy by electronic collisions and Auger processes and beeome bound into a Bohr orbit , usually the lowest S state , about a nucleus. Thus any discussion of the subsequent nuclear disintegration can conveniently start from the ground state of the mesic nuclear atom.

The enormously complicated character of the field equations makes exact calculation impossible , and so certain approximations are required. The oldest and simplest device is perturbation theory based on the weak coupling approximation. This method was used by Yukawa in his original paper and only recently have doubts been cast upon its use in calculations of nuclear forces. It has been shown by Levy (1952) , using a non-adiabatic treatment , that the symmetrical pseudoscalar meson theory gives a proton-neutron force which is repulsive at short distances. This startling result seems to be due to the fact that relativistic effects and the recoil of the nucleons are taken into account. However , it should be borne in mind that the nuclear force is at least a second-order effect involving the emission and absorption of one or more virtual mesons , and is thus connected with the self-energy divergences that beset all field theories. The absorption of a meson by a nucleus is much simpler in that there is a first-order contribution which may in fact be the most important part. In what follows we use the conventional first-order perturbation theory as the simplest way of obtaining qualitative results , even though the convergence of the theory is in no way assured.

According to first-order perturbation theory , the transition probability leading to a certain type of disintegration is proportional to

$$P = \int w \, d\rho_E \tag{1}$$

where $d\rho_E$ is the number of final states per unit energy and the integration is over all final states which satisfy the energy and momentum conservation laws. We have put

$$w = \sum |H|^2 \tag{2}$$

where the summation goes over all spin states , H being the matrix element of the perturbing hamiltonian with respect to initial anf final states :

$$H = \int \Psi_F^* \, G' \, \Psi_I \tag{3}$$

Here Ψ_I and Ψ_F are the wave functions describing the initial and final states of the nucleons and G' is an operator whose form depends on the kind of meson theory assumed.

In order to calculate H from (3) , the meson field should be quantized to represent charged mesons in the coulomb field of the nucleus ; it would then be described by the occupation numbers of negative mesons in the various discrete levels together with the occupation densities of positive and negative mesons in the levels of the continuous spectrum. The meson field was treated in this way by Snyder and Weinberg (194o) , but in our case a simplifying approximation is permissible. The operator G' can be written as

$$G' = \sum_p \delta(\underset{\sim}{r} - \underset{\sim}{r}_p)\, \Gamma_p'\, \psi(\underset{\sim}{r}) \tag{4}$$

where ψ is the meson wave function , the summation goes over all protons in the nucleus , and Γ_p' is the operator which destroys a meson and converts a proton p into a neutron. In momentum space we have

$$G_p' = \Gamma_p'\, \psi(\underset{\sim}{r}) = \Gamma_p' \int e^{i \underset{\sim}{r} \cdot \underset{\sim}{k}}\, \chi(\underset{\sim}{k})\, d\underset{\sim}{k} \tag{5}$$

The matrix element of G_p' corresponding to destruction of a meson is thus

$$G_p = \Gamma_p \sqrt{\lambda} \int e^{i \underset{\sim}{r} \cdot \underset{\sim}{k}}\, \chi(\underset{\sim}{k})\, (\lambda^2 + k^2)^{-\frac{1}{4}}\, d\underset{\sim}{k} \tag{6}$$

where λ is the inverse Compton wavelength of the meson and the operator Γ_p now only involves the nucleon variables. Since the wave function ψ represents the meson in the lowest Bohr orbit , only wave numbers k, $0 < k < \frac{Zme^2}{\hbar^2} = \frac{Z\lambda}{137}$ contribute to the integral in (6) , where Ze is the nuclear charge. The last factor in the integrand is nearly constant , so we can replace (6) by

$$G_p = \Gamma_p \, \psi(\underset{\sim}{r}) \tag{7}$$

For mesons of spin zero , either scalar (S) or pseudoscalar (PS) theory is required. If the nucleons are treated as non-relativistic particles , in the former case only scalar coupling , and in the latter case only pseudovector coupling , contribute to the interaction , giving for G_p the following expressions :

$$G_p = \begin{cases} \gamma_p & \text{(S)} \\ \gamma_p \, \underset{\sim}{\sigma}_p \cdot \underset{\sim}{\nabla} & \text{(PS)} \end{cases} \tag{8}$$

where γ_p is the operator that turns a proton p into a neutron and $\underset{\sim}{\sigma}_p$ is the Pauli spin operator of this proton. The non-relativistic approximation should be valid since the maximum energy of a nucleon after the disintegration is about 9o MeV , although at this energy non-static and relativistic effects will presumably be starting to make themselves felt.

Since the binding energy of the meson in the lowest Bohr level is only 3keV , the non-relativistic wave function can be used for ψ . We cannot , however , set $\psi = e^{-r/a}$ with $a = \frac{\hbar^2}{Zme^2}$, because this wave function has a discontinuity in $\partial\psi/\partial r$ at $r=0$, corresponding to a point nucleus. In fact , the spread of charge over the nucleus modifies the wave function near $r=0$ so that $\left.\frac{\partial\psi}{\partial r}\right|_{r=0} = 0$. In terms of a series expansion we must have

$$\psi = A - Br^2 \quad + \text{higher terms} \qquad (9)$$

Only the region near $r=0$ contributes to the matrix element (3) , so the higher terms in (9) can be neglected. It turns out that the permitted angular momenta of the emitted particles is severely limited by the neglect of these higher terms. In all cases considered , the wave functions are such that tha first term in (9) gives no contribution. These points are discussed further in the next section on the symmetry properties of the wave functuons.

2. 2. Wave Functions for Bound States

Since the exact form of the nuclear forces is not known , the correct nuclear wave functions cannot be used. Even if some plausible nuclear forces are assumed the corresponding Schrödinger equation is too complicated to be solved. It is therefore necessary to guess a simple form of the wave function which has roughly the right physical characteristics. For heavy nuclei the independent particle model is often used , but this must be rejected for He^4 where the mutual correlations of the nucleons may be important.

All other wave functions that have been suggested involve the complete separation of space and spin coordinates. This is a very natural assumption to make and is almost certainly not true. It is retained here since to do otherwise would make for great complexity unwarranted by the few hard facts about nuclei that are known. In addition the high degree of symmetry of the He^4 nucleus gives an air of plausibility to this assumption. Let particles 1 and 2 be neutrons , 3 and 4 protons and let $\underset{\sim}{r}_{ij} = \underset{\sim}{r}_i - \underset{\sim}{r}_j$, where $\underset{\sim}{r}_i$ is the position of the i'th nucleon. Then we put

$$\Psi_I = \frac{1}{2}(\alpha_1\beta_2 - \beta_1\alpha_2)(\alpha_3\beta_4 - \beta_3\alpha_4)\,\psi_I \tag{1o}$$

where α_i and β_i are the Pauli spin functions of the i'th nucleon. Since the spin factor in (1o) is antisymmetric in neutrons and protons , the space part ψ_I must be symmetric. It is convenient to make ψ_I symmetric in all pairs of nucleons , a practice that may perhaps be justified on the assumption of charge-independent nuclear forces.

In a study of the binding energies of light , Fröhlich et. al. (1947) used for ψ_I the form of a product of deuteron wave functions :

$$\psi_I = e^{-\alpha \sum_{i<j}^{4} |r_{ij}|} \tag{11}$$

the parameter α being determined by a variational method. If the nuclear forces are mainly two-body interactions , this may be quite a good wave function , but unfortunately there is no known case where the matrix element (3) can be evaluated with this wave function. Even in their relatively simple case , Fröhlich et. al. had to resort to laborious numerical integrations. It was felt that the great amount of numerical integration required by the use of these wave functions in the present context was not justified by the qualitative nature of the results.

We are left with two possibilities. The first of these is the ubiquitous Gaussian wave function :

$$\psi_I = e^{-\alpha \sum_{i<j} r_{ij}^2} \tag{12}$$

which has the great virtue that all matrix elements are very easily evaluated. It has , however , the drawback that the parameter α is not well-defined , and the results of calculation often depend critically on α . The second possibility is connected with the asymptotic form of the "correct" wave function. This must satisfy the free-particle Schrödinger equation :

$$\sum_{i=1}^{4} \nabla_i^2 \psi_I - \beta^2 \psi_I = 0 \qquad (13)$$

Putting

$$X^2 = \frac{1}{4} \sum_{i<j}^{4} r_{ij}^2 \qquad (14)$$

we try for ψ_I as a function only of X . It is easily seen that the solution which vanishes at infinity is $X^{-7/2} K_{7/2}(\beta X)$, where $K_{7/2}$ is a Hankel function. Taking the asymptotic form of this expression for large X we get

$$\psi_I = X^{-n} e^{-\beta X} \qquad (15)$$

with $n=4$. This form of the wave function has been used by Irving (1951) , who took β and n as variational parameters. This seems to be the most satisfactory way of using this function , since $n=4$ gives a pole at $X=0$ of such high order that the function cannot be normalized. In practice we put $n=1$ in order to make the integral (3) manageable ; the results are close to those obtained from Irving's best value of $n=\frac{1}{4}$. The parameter β is fixed by the binding energy B :

$$\beta^2 = \frac{2MB}{\hbar^2} \qquad (16)$$

where M is the nucleon mass.

At this stage we do not wish to commit ourselves to the use of either (15) or (18) as initial wave function we merely assume that

$$\psi_I = \phi(X) \qquad \text{L (17)}$$

and leave the form of the function ϕ open.

2. 3. <u>Wave Functions for Final States</u>

In this case also it is necessary to assume a reasonable form for the wave functions. Since the "correct" wave functions extend to infinity , the asymptotic form may be taken as valid over the whole range. This is equivalent to neglecting the effect of the nuclear forces , and so should lead to reasonable results provided the emitted particles are energetic enough. In all the cases considered here the average energy per particle after the disintegration is about 3o MeV., which is high enough to justify the approximation. The only case of meson capture where the nuclear forces need be taken into account is the reaction

$$H^2 + \pi^- \rightarrow 2n + \gamma \qquad (18)$$

where the γ-ray spectrum depends critically on the final-state wave function.

The importance of angular momentum and parity considerations in the case of meson capture by deuterium suggests that it may be worth while to choose final-state wave functions in which these properties are exhibited directly. The commonly used wave functions involving plane waves do not satisfy this requirement , although they explicitly account for conservation of momentum , another necessary condition.

Since momentum and angular momentum are non-commuting observables , the construction of wave functions which represent particles having definite values of both is in general not possible . In our case , the total momentum is zero , and it is shown in the next section how the required wave functions may be obtained by using relative coordinates.

After meson capture , He^4 may disintegrate in three ways , giving a proton , deuteron or triton with three , two and one neutrons respectively. It is convenient now to consider these processes in more detail , starting with proton emission.

3. Proton Emission from He^4

3. 1. Relative Coordinates and Wave Functions

If the initial wave function is given by (1o), then either particle 3 or 4 will be a proton in the final state. Because of the symmetry of the interaction we need allow for only one of these cases Thus let the nucleons have positions $\underset{\sim}{r}_i$, where 1 , 2 and 3 refer to neutrons and 4 is a proton . We put

$$\sum_1^4 \underset{\sim}{r}_i = 0 \tag{1}$$

and define relative coordinates $\underset{\sim}{x}_i$ by

$$\begin{aligned} \underset{\sim}{x}_1 &= \frac{1}{\sqrt{2}}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \\ \underset{\sim}{x}_2 &= \sqrt{\frac{2}{3}}\left(\frac{1}{2}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2) - \underset{\sim}{r}_3\right) \\ \underset{\sim}{x}_3 &= \frac{\sqrt{3}}{2}\left(\frac{1}{3}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3) - \underset{\sim}{r}_4\right) \end{aligned} \tag{2}$$

The Schrödinger equation for the free particles is now

$$(\nabla_1^2 + \nabla_2^2 + \nabla_3^2 + k^2)\Psi_F = 0 \tag{3}$$

where the differentiations refer to the x-coordinates and

$$k^2 = \frac{2ME_1}{\hbar^2} \tag{4}$$

Here E_1 is the total kinetic energy in the final state , about 112 MeV .

It is easily seen that the total orbital angular momentum operator in the new coordinates is

$$\underset{\sim}{\mathcal{M}} = -i\hbar \sum_{i=1}^{3} \underset{\sim}{x}_i \times \underset{\sim}{\nabla}_i \tag{5}$$

This is the same form that $\underset{\sim}{\mathcal{M}}$ takes in the r-coordinates and shows that if (3) is solved by separating the variables , giving Ψ_F as a product of spherical harmonics and radial factors , these can be interpreted in the usual way as representing various angular momenta.

Let

$$U_{l\,m}(x_i) = Y_l^m(\theta_i \varphi_i)\, f_l(\lambda_i x_i) \tag{6}$$

where Y_l^m is a normalized spherical harmonic , φ_i and φ_i the polar angles of $\underset{\sim}{x}_i$ and

$$f_l(\lambda x) = \sqrt{\frac{\pi\lambda}{Rx}}\; J_{l+\frac{1}{2}}(\lambda x) \tag{7}$$

is the usual radial wave function , normalized in a large sphere of radius R . The λ_i are separation parameters which determine the way in which the energy is distributed beteeen the particles . They satisfy

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = k^2 \tag{8}$$

The appropriate final state wave functions are then constructed out of the U_ℓ^m together with the spin functions α_i and β_i, in such a way that the total angular momentum vanishes and the parity has the correct sign. It is also necessary to make the wave function antisymmetrical in the coordinates of all three neutrons. To this end we find a normalized wave function $\psi_F(1234)$ which is antisymmetric in neutrons 1 and 2 only. Then the required wave function is

$$\Psi_F(1234) = \psi_F(1234) + \psi_F(2314) + \psi_F(3124). \tag{9}$$

The use of numerals as arguments of a function in (9) means the space and spin coordinates of the corresponding particles.

The number of final states that need be considered is limited by the fact that only $\ell = 0$ and $\ell = 1$ give contributions; this is a consequence of the form

$$\psi = r^2 \tag{10}$$

(cf. equation (2. 9) et seq.) which is assumed for the meson wave function.

For brevity we define the following combinations of spin functions which are of frequent occurence :

$$A_{ij} = \frac{1}{\sqrt{2}} (\alpha_i \beta_j - \beta_i \alpha_j) \tag{11}$$

$$\left.\begin{aligned} S^{1}_{ij} &= \alpha_i \alpha_j \\ S^{0}_{ij} &= \frac{1}{\sqrt{2}} (\alpha_i \beta_j + \beta_i \alpha_j) \\ S^{-1}_{ij} &= \beta_i \beta_j \end{aligned}\right\} \tag{12}$$

$$K_{ijlm} = \frac{1}{\sqrt{3}} \sum_{\lambda=-1}^{1} (-)^{\lambda} S^{\lambda}_{ij} S^{-\lambda}_{lm} \tag{13}$$

We also need

$$Q(\underset{\sim}{x}_i, \underset{\sim}{x}_j) = \frac{1}{\sqrt{3}} \sum_{m=-1}^{1} (-)^{m} U_1^{m}(x_i)\, U_1^{-m}(x_j) \tag{14}$$

We now require certain linear combinations of these functions with the U_l^m. The coefficients may be found by a simple application of group theory, and have been tabulated by Condon and Shortley (1951).

Assuming that the initial nucleus has even parity , the final state wave function must have even or odd parity according as the meson is scalar or pseudoscalar. The only wave functions ψ_F needed are listed below.

Expressions for ψ_F (1234) :

Scalar Meson

(a) $$A_{12}\ A_{34}\ U_0(1)\ U_0(2)\ U_0(3) \tag{15a}$$

(b) $$A_{12}\ A_{34}\ U_0(1)\ Q(23) \tag{15b}$$

(c) $$K_{1234}\ U_0(3)\ Q(12) \tag{15c}$$

(d) $$K_{1234}\ U_0(2)\ Q(13) \tag{15d}$$

Pseudoscalar Meson

(a) $$\frac{1}{\sqrt{3}}\ A_{12}\ U_0(1)\ U_0(3) \sum_{m=-1}^{1} (-)^m\ U_1^m(2)\ S_{34}^{-m} \tag{16a}$$

(b) $$\frac{1}{\sqrt{3}}\ A_{34}\ U_0(2)\ U_0(3) \sum_{m=-1}^{1} (-)^m\ U_1^m(1)\ S_{12}^{-m} \tag{16b}$$

(c) $$\frac{1}{\sqrt{6}}\ U_0(2)\ U_0(3) \begin{vmatrix} U_1^1(1) & U_1^0(1) & U_1^{-1}(1) \\ S_{12}^1 & S_{12}^0 & S_{12}^{-1} \\ S_{34}^1 & S_{34}^0 & S_{34}^{-1} \end{vmatrix} \tag{16c}$$

The second and third terms in (9) are obtained by permuting the space and spin coordinates. Let the permutations

$$(\underset{\sim}{r_1}, \underset{\sim}{r_2}, \underset{\sim}{r_3}) \rightarrow (\underset{\sim}{r_2}, \underset{\sim}{r_3}, \underset{\sim}{r_1}) \longrightarrow (\underset{\sim}{r_3}, \underset{\sim}{r_1}, \underset{\sim}{r_2})$$

induce the transformatoons

$$\underset{\sim}{x}_i \rightarrow \underset{\sim}{y}_i \rightarrow \underset{\sim}{z}_i \tag{17}$$

Then the interaction operators , $\gamma_3 r_3^2$ (scalar) and $\gamma_3\ \underset{\sim}{\sigma}_3 \cdot \underset{\sim}{r}_3$ (pseudoscalar) may be expressed in terms of $\underset{\sim}{r}_3$ given by

$$\begin{aligned} \underset{\sim}{r}_3 &= \frac{1}{2\sqrt{3}}\ \underset{\sim}{x}_3 - \sqrt{\frac{2}{3}}\ \underset{\sim}{x}_2 \\ &= \frac{1}{2\sqrt{3}}\ \underset{\sim}{y}_3 + \frac{1}{\sqrt{6}}\ \underset{\sim}{y}_2 - \frac{1}{\sqrt{2}}\ \underset{\sim}{y}_1 \\ &= \frac{1}{2\sqrt{3}}\ \underset{\sim}{z}_3 + \frac{1}{\sqrt{6}}\ \underset{\sim}{z}_2 + \frac{1}{\sqrt{2}}\ \underset{\sim}{z}_1 \end{aligned} \tag{18}$$

3. 2. Matrix Elements

With Ψ_I given by (2. lo) and (2. 17) the matrix elements are readily evaluated. For example , (15a) gives a matrix element

$$\begin{aligned} H = &\int \phi(X)\, r_3^2\, U_0(x)\, dx \\ &-\tfrac{1}{2}\int \phi(X)\, r_3^2\, U_0(y)\, dy \\ &-\tfrac{1}{2}\int \phi(X)\, r_3^2\, U_0(z)\, dz \end{aligned} \tag{19}$$

where $U_0(x)$ means $U_0(x_1)\ U_0(x_2)\ U_0(x_3)$

and dx means $d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2\, d\underset{\sim}{x}_3$, and so on. Expanding r_3^2 and collecting terms gives , after integrating over the angles ,

$$H = \frac{1}{2}(4\pi)^{3/2} \iiint_0^\infty \phi(X)(x_2^2 - x_1^2)\, f_0(\lambda_1 x_1)\, f_0(\lambda_2 x_2) \times f_0(\lambda_3 x_3)\, x_1^2 x_2^2 x_3^2\, dx_1\, dx_2\, dx_3 \tag{20}$$

Since

$$X^2 = x_1^2 + x_2^2 + x_3^2 \tag{21}$$

we transform to polar coordinates in the $x_1 x_2 x_3$ space , and use two theorems on Bessel functions (Watson 1944 , pp 373 , 376) , namely

$$\int_0^{\pi/2} J_\mu(z \sin\theta) \sin^{\mu+1}\theta \cos^{2\nu+1}\theta\, d\theta = \frac{2^\nu \Gamma(\nu+1)}{z^{\nu+1}} J_{\mu+\nu+1}(z) \tag{22}$$

and

$$\int_0^{\pi/2} J_\mu(z\sin\theta)\, J_\nu(Z\cos\theta) \sin^{\mu+1}\theta \cos^{\nu+1}\theta\, d\theta = \frac{z^\mu Z^\nu J_{\mu+\nu+1}\left(\sqrt{Z^2+z^2}\right)}{(Z^2+z^2)^{\frac{1}{2}(\mu+\nu+1)}} \tag{23}$$

to get finally , putting

$$\Phi_1 = \frac{4\pi^{3/2}}{k^{11/2}} \int_0^\infty \phi(t)\, J_{\frac{11}{2}}(kt)\, t^{13/2}\, dt \, , \tag{24}$$

$$H_a(S) = \lambda_1 \lambda_2 \lambda_3 (\lambda_1^2 - \lambda_2^2) \left(\frac{\pi}{R}\right)^{\frac{3}{2}} \Phi_1 \tag{25a}$$

In the same way are found the remaining matrix elements corresponding to (15) :

$$H_b(S) = -\sqrt{\frac{2}{3}}\, \lambda_1 \lambda_2^2 \lambda_3^2 \left(\frac{\pi}{R}\right)^{3/2} \Phi_1 \tag{25b}$$

$$H_c(S) = -\frac{2}{\sqrt{3}}\, \lambda_1^2 \lambda_2^2 \lambda_3 \left(\frac{\pi}{R}\right)^{3/2} \Phi_1 \tag{25c}$$

$$H_d(S) = -\sqrt{\frac{2}{3}}\, \lambda_1^2 \lambda_2 \lambda_3^2 \left(\frac{\pi}{R}\right)^{3/2} \Phi_1 \tag{25d}$$

The matrix elements in the pseudoscalar case , corresponding to (16) are

$$H_a(PS) = -\lambda_1 \lambda_2^2 \lambda_3 \left(\frac{\pi}{R}\right)^{3/2} \Phi_2 \tag{26a}$$

$$H_b(PS) = \frac{1}{\sqrt{3}}\, \lambda_1^2 \lambda_2 \lambda_3 \left(\frac{\pi}{R}\right)^{3/2} \Phi_2 \tag{26b}$$

$$H_c(PS) = \sqrt{\frac{2}{3}}\, \lambda_1^2 \lambda_2 \lambda_3 \left(\frac{\pi}{R}\right)^{3/2} \Phi_2 \tag{26c}$$

where we have put

$$\Phi_2 = \frac{4\sqrt{6}\,\pi^{3/2}}{k^{9/2}} \int_0^\infty \phi(t)\, J_{9/2}(kt)\, t^{11/2}\, dt . \tag{27}$$

From (25) and (26), $W = \sum |H|^2$ has the following values for proton emission :

$$W_p(S) = \left(\frac{\pi}{R}\right)^3 |\Phi_1|^2 \lambda_1^2 \lambda_2^2 \lambda_3^2$$

$$\times \left(\lambda_1^4 + \lambda_2^4 - \frac{2}{3}\lambda_1^2\lambda_2^2 + \frac{2}{3}\lambda_2^2\lambda_3^2 + \frac{2}{3}\lambda_3^2\lambda_1^2\right) \tag{28}$$

$$W_p(PS) = \left(\frac{\pi}{R}\right)^3 |\Phi_2|^2 \lambda_1^2 \lambda_2^2 \lambda_3^2 (\lambda_1^2 + \lambda_2^2) \tag{29}$$

3. 3. Transition Probabilities and Energy Spectra of Protons

The transition probabilities can now be obtained from W , according to (2. 1). From the asymptotic form of the wave functions $f_e(\lambda x)$ we see that there are a number

$$dn = \frac{R}{\pi} d\lambda \tag{30}$$

of wave functions corresponding to the small range $d\lambda$. It follows that the transition probability $\frac{2\pi}{\hbar} P$ is given by

$$P = \frac{1}{\Delta E}\left(\frac{R}{\pi}\right)^3 \int_{\Delta E} W \, d\lambda_1 d\lambda_2 d\lambda_3 \tag{31}$$

where the integration extends over an energy shell of thickness ΔE. (31) is equivalent to

$$P = \left(\frac{R}{\pi}\right)^3 \frac{d}{dE} \int W \, d\lambda_1 \, d\lambda_2 \, d\lambda_3 \tag{32}$$

the integration now extending over the whole of the λ-space inside this shell. The equation of the shell is given by the energy-conservation relation (8).

All the wave functions (15) and (16) represent states in which the proton has a definite energy $\epsilon = 3\hbar^2 \lambda_3^2 / 8M$; thus the energy spectrum of the proton may be found by omitting the integration over λ_3 in (32). The integrations are elementary and give the following expressions for the transition probabilities :

$$P_p(S) = \frac{\pi}{1155} \cdot \frac{1}{E_1} \left(\frac{2ME_1}{\hbar^2}\right)^{\frac{13}{2}} |\Phi_1|^2 \tag{33}$$

$$P_p(PS) = \frac{\pi}{630} \cdot \frac{1}{E_1} \cdot \left(\frac{2ME_1}{\hbar^2}\right)^{\frac{11}{2}} |\Phi_2|^2 \tag{34}$$

The corresponding proton energy spectra are

$$dN_p(S) = \left(\epsilon_0 + \frac{\epsilon}{3}\right)\left(\epsilon_0 - \epsilon\right)^3 \epsilon^{\frac{1}{2}} \, d\epsilon \tag{35}$$

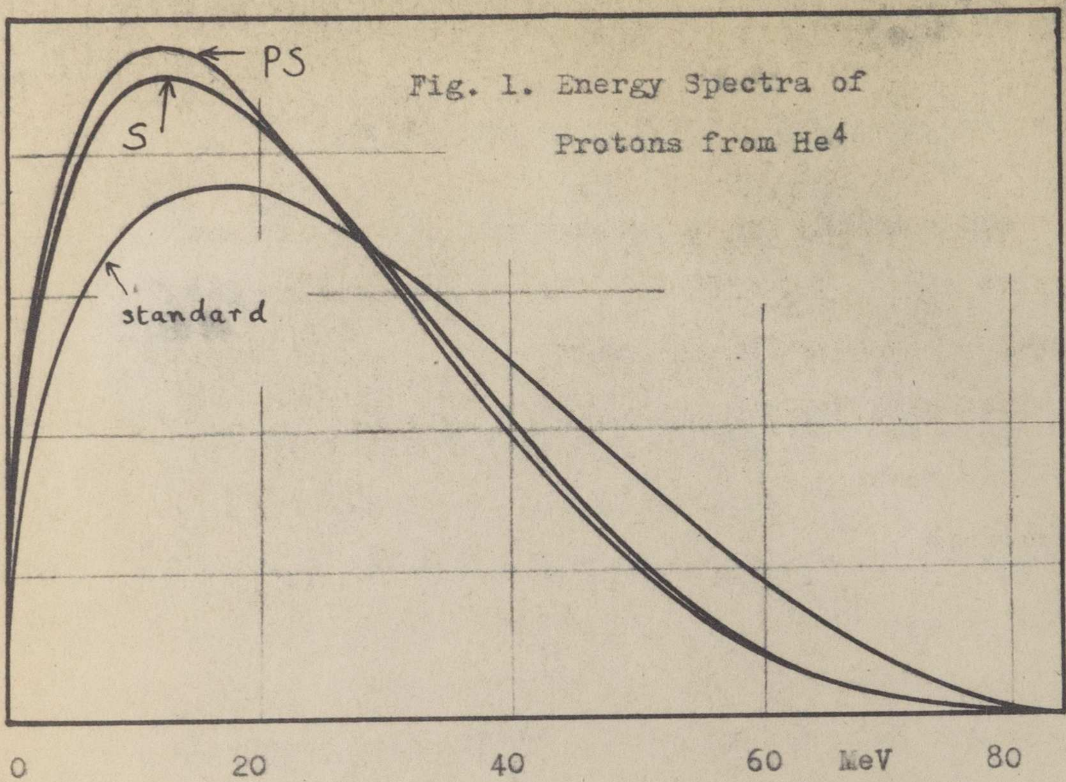


Fig. 1. Energy Spectra of Protons from He^4

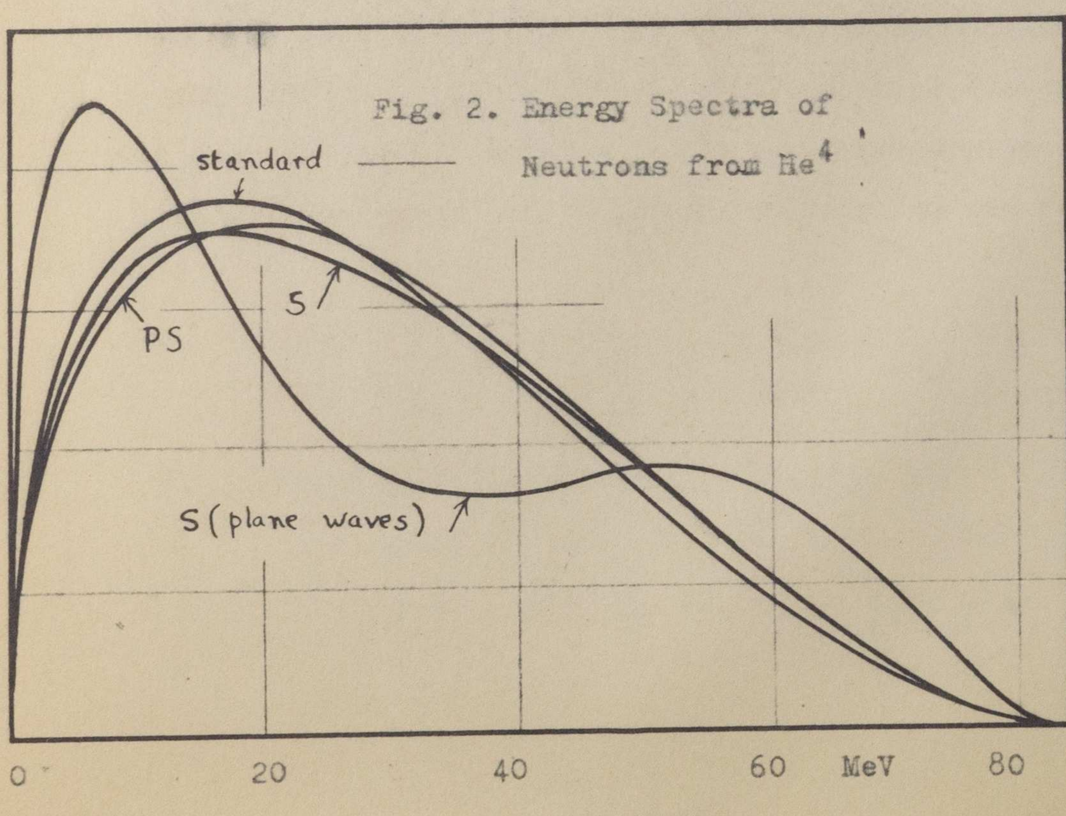


Fig. 2. Energy Spectra of Neutrons from He^4

$$dN_p(PS) = (\epsilon_0 - \epsilon)^3 \epsilon^{1/2}\, d\epsilon \tag{36}$$

where ϵ is the proton energy and ϵ_0 the maximum value of ϵ, about 84 MeV. These expressions for dN_p show only the dependence on ϵ, all constant multiplying factors having been omitted.

Putting $W = \lambda_1^2 \lambda_2^2 \lambda_3^2$ in (32), one gets a spectrum

$$dN_p' = (\epsilon_0 - \epsilon)^2 \epsilon^{1/2}\, d\epsilon \tag{37}$$

which is independent of the meson interaction. It is convenient to take this as a standard spectrum with which to compare (35) and (36) in order to see the extent to which the meson interaction affects the proton spectrum. Figure 1. shows the proton spectra in the scalar, pseudoscalar and "standard" cases

3. 4. Energy Spectra of Neutrons

The energy spectrum of the neutrons emitted by He^4 is an observable quantity that may depend more on the meson interaction than the corresponding proton spectrum. This is because the meson is supposed to interact only with a single nucleon , which is emitted as a neutron.

The calculation of the neutron energy spectra is not so straightforward as the proton spectra because the wave functions (15) and (16) do not represent states in which the neutrons have definite energy. We must thus first find expressions for the neutron spectrum represented by each of these wave functions . This can conveniently be done in momentum space. It turns out that all the wave functions (15) and (16) give the same result , so we consider only (15a) , which may be written , disregarding a constant factor , as

$$\psi_F = \frac{1}{\lambda_1 \lambda_2 \lambda_3} A_{12} A_{34} \int e^{-i \sum_1^3 \underset{\sim}{x_i} \cdot \underset{\sim}{k_i}} \times G(\lambda_j ; \underset{\sim}{k_j}) \, d\underset{\sim}{k_1} \, d\underset{\sim}{k_2} \, d\underset{\sim}{k_3} \qquad (38)$$

with

$$G(\lambda_j ; \underset{\sim}{k}_j) = \prod_{i=1}^{3} \delta(\lambda_i - |\underset{\sim}{k}_i|) \tag{39}$$

Regarding (38) as a superposition of plane waves , we have

$$\frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} |G|^2 d\underset{\sim}{k}_1 \, d\underset{\sim}{k}_2 \, d\underset{\sim}{k}_3$$

as the relative probability of a state with wave-numbers $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$ and $\underset{\sim}{k}_3$. The neutron momentum distribution is thus

$$F_j(\underset{\sim}{p}) \, d\underset{\sim}{p} = \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \int' |G|^2 d\underset{\sim}{k}_1 \, d\underset{\sim}{k}_2 \, d\underset{\sim}{k}_3 \tag{4o}$$

where the prime on the integral sign means that the integration over the neutron momentum $\underset{\sim}{p}$ is not carried out. There are three expressions like (4o) corresponding to the three neutrons j emitted.

Since G is a product of δ-functions , $F_j(\underset{\sim}{p})$ given by (4o) is infinite. The reason for this is that we have tacitly gone over to an infinite region of normalization in coordinate space for the wave function , instead of a large sphere R as before. The wave function is now normalized to an infinite quantity , in fact to $\delta(o)$. The solution of this difficulty is to replace $|G|^2$ in (40) by $G^*(\lambda_j' ; k_j)\, G(\lambda_j ; k_j)$. Then a factor $\prod_1^3 \delta(\lambda_j - \lambda_j')$ may be extracted from the integrand. With $\lambda_j = \lambda_j'$, this factor is just the infinite normalization constant. We have finally

$$F_j(p)\, dp = \frac{1}{\lambda_1^2 \lambda_2^2 \lambda_3^2} \int' G(\lambda_i ; k_i)\, dk_1\, dk_2\, dk_3 \tag{41}$$

According to (38) , we must have

$$\sum_1^3 x_i \cdot k_i = \sum_1^4 r_i \cdot p_i \tag{42}$$

where the p_i are the wave numbers of the four nucleons. From the definition of the relative coordinates (2) and from (42) we find

$$\begin{aligned} k_1 &= -\sqrt{2}\left(p_2 + \tfrac{1}{2} p_3 + \tfrac{1}{2} p_4\right) \\ k_2 &= -\sqrt{\tfrac{3}{2}}\left(p_3 + \tfrac{1}{3} p_4\right) \\ k_3 &= -\frac{2}{\sqrt{3}}\, p_4 \\ \sum_1^4 p_i &= 0 \end{aligned} \tag{43}$$

We now easily find that

$$F_1(\underset{\sim}{p}) = F_2(\underset{\sim}{p})$$

$$= \frac{1}{2\lambda_1^2\lambda_2^2\lambda_3^2}\int \delta\left(\frac{1}{\sqrt{2}}\lambda_1 - |\underset{\sim}{p} + \tfrac{1}{2}\underset{\sim}{k}_3 + \tfrac{1}{2}\underset{\sim}{k}_4|\right)$$

$$\times\, \delta\left(\sqrt{\tfrac{2}{3}}\lambda_2 - |\underset{\sim}{k}_3 + \tfrac{1}{3}\underset{\sim}{k}_4|\right)\delta\left(\frac{\sqrt{3}}{2}\lambda_3 - |\underset{\sim}{k}_4|\right) d\underset{\sim}{k}_3\, d\underset{\sim}{k}_4 \tag{44}$$

$$F_3(\underset{\sim}{p}) = \frac{\pi}{\lambda_2^2\lambda_3^2}\int \delta\left(\sqrt{\tfrac{2}{3}}\lambda_2 - |\underset{\sim}{p} + \tfrac{1}{3}\underset{\sim}{k}_4|\right)$$

$$\times\, \delta\left(\frac{\sqrt{3}}{2}\lambda_3 - |\underset{\sim}{k}_4|\right) d\underset{\sim}{k}_4 \tag{45}$$

The functions F_j are here all normalized to the same value which is independent of the λ_i .

The momentum distribution of the neutrons emitted in the disintegration is now simply

$$dN_n = \frac{d}{dE}\int W \sum_{j=1}^{3} F_j(\underset{\sim}{p})\, d\lambda_1\, d\lambda_2\, d\lambda_3 \tag{46}$$

with W given by (28) and (29). The simplest way to calculate (46) is to insert the integral representations (44) and (45) for the F_j and change the order of integration . After some tedious calculation we find the distribution of the neutron energy ϵ to be

$$dN_n(S) = \left(57 - 22\frac{\epsilon}{\epsilon_0} + 61\left(\frac{\epsilon}{\epsilon_0}\right)^2\right)$$
$$\times (\epsilon_0 - \epsilon)^2 \epsilon^{\frac{1}{2}} d\epsilon \tag{47}$$

$$dN_n(PS) = \left(5 + 3\frac{\epsilon}{\epsilon_0}\right)(\epsilon_0 - \epsilon)^2 \epsilon^{\frac{1}{2}} d\epsilon \tag{48}$$

In Figure 2. the neutron energy spectra in the scalar and pseudoscalar cases are compared with the standard spectrum , which of course is the same standard spectrum as for the protons, and also with the spectrum obtained in the scalar case when plane waves are used to represent final states (Clark and Ruddlesden 1951).

3. 5 Angular Distribution of Neutrons

The formalism of the last section can be applied without much change to calculate the angular distribution of the emitted neutrons. A possible experiment might measure the distribution in the angle between the directions of emission of the proton and any neutron , and we restrict attention to this case.

Corresponding to (41) we have the angular distribution of the j'th neutron as

$$P_j(\mu_j)\,d\mu_j = \frac{1}{\lambda_1^2\lambda_2^2\lambda_3^2}\int' G\, d\underset{\sim}{k}_1\, d\underset{\sim}{k}_2\, d\underset{\sim}{k}_3 \tag{49}$$

where μ_j is the cosine of the angle between $\underset{\sim}{p}_j$ and $\underset{\sim}{p}_4$, and the prime on the integral sign means that the integration does not extend over μ_j . The observed angular distribution $H(\mu)\,d\mu$ is then given by (46) with F_j replaced by P_j .

Consider for simplicity the "standard" case with $W = \lambda_1^2\lambda_2^2\lambda_3^2$. We easily find that

$$H_0(\mu)\,d\mu = \frac{d}{dE}\int' \lambda_1^2\, d\lambda_1\, d\underset{\sim}{p}\, d\underset{\sim}{k}_4 \tag{50}$$

the region of integration being

$$\lambda_1^2 + \frac{3}{2}\left(\underset{\sim}{p} + \frac{1}{3}\underset{\sim}{k}_4\right)^2 + \frac{4}{3}\underset{\sim}{k}_4^2 < k^2 \tag{51}$$

In (50) , μ is the cosine of the angle betwween $\underset{\sim}{p}$ and $\underset{\sim}{k}_4$. Integration over λ_1 gives

$$H_0(\mu) = \frac{d}{dE}\int\left(k^2 - \frac{3}{2}p^2 - \frac{3}{2}k_4^2 - p k_4 \mu\right)^{3/2} \times p^2 k_4^2 \, dp \, dk_4 \tag{52}$$

The substitution

$$\left.\begin{aligned} p &= k x^{1/2}\cos\theta \\ k_4 &= k\, x^{1/2}\sin\theta \end{aligned}\right\} \tag{53}$$

now gives

$$H_0(\mu) = \frac{d}{dE} k^9 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta \times \int_0^{2/(3+\mu\sin 2\theta)} \left(1 - \frac{x}{2}(3+\mu\sin 2\theta)\right)^{3/2} x^2 \, dx \tag{54}$$

A factor containing E splits off and the second integral can be evaluated , leaving , apart from a

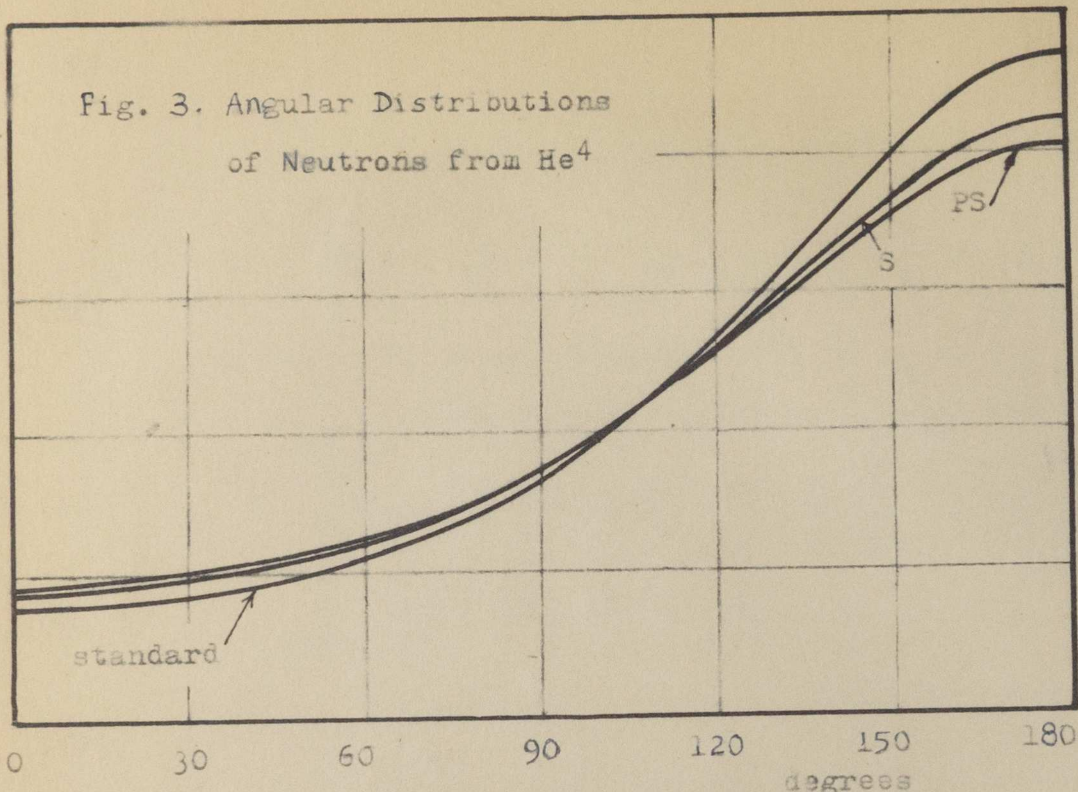
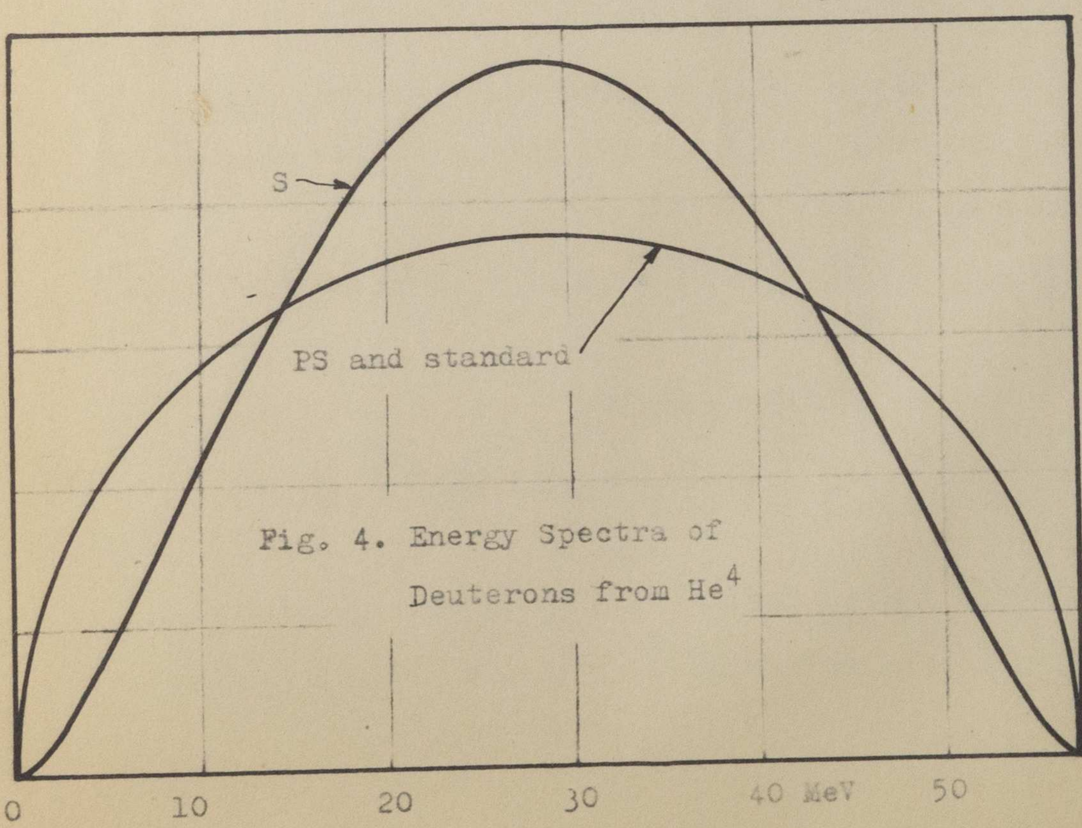


Fig. 3. Angular Distributions of Neutrons from He^4
PS
S
standard
0
30
60
90
120
150
180
degrees



S
PS and standard
Fig. 4. Energy Spectra of Deuterons from He^4
0
10
20
30
40 MeV
50

constant factor ,

$$H_0(\mu) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cos^2\theta \, d\theta}{(3 + \mu \sin 2\theta)^3} \tag{55}$$

Putting $t = \tan\theta$ gives

$$H_0(\mu) = \int_0^{\infty} \frac{t^2 \, dt}{(1 + \frac{2}{3}\mu t + t^2)^3} \tag{56}$$

$$= \frac{27}{8\varpi^5}(9 + 2\mu^2)\cos^{-1}\frac{\mu}{3} - \frac{81}{8}\frac{\mu}{\varpi^4} \tag{57}$$

where $\varpi = \sqrt{9-\mu^2}$.

The calculations for the scalar and pseudoscalar cases are made in the same by using the expressions (28) and (29) for W , to give

$$H_s(\mu) = \frac{27}{\varpi^7}(1521 + \mu^2 - 50\mu^4)\cos^{-1}\mu/3 - \frac{\mu}{\varpi^6}(11673 - 2137\mu^2) \tag{58}$$

and

$$H_{PS}(\mu) = \frac{9}{\varpi^7}(171 - 5\mu^2 - 6\mu^4)\cos^{-1}\frac{\mu}{3}$$

$$- \frac{\mu}{3\varpi^6}(1251 - 259\mu^2) \qquad (59)$$

The angular correlation functions of the neutrons in the scalar , pseudoscalar and "standard" cases , normalized to the same value , are shown in Figure 3.

4. Other Reactions in He^4

4. 1. Deuteron Emission

The next reaction to be considered is

$$He^4 + \pi^- \rightarrow H^2 + 2n \tag{1}$$

in which we are interested in the transition probability and the energy spectrum of the emitted deuteron. We expect this reaction to be much less probable than proton emission because a smaller number of momentum states is allowed in the integral (2. 1). Accordingly the neutron spectrum need not be considered.

We assume that the deuteron wave function can be separated into space and spin parts ,

$$\psi_d^m(3,4) = S_{34}^m \, \chi(\underset{\sim}{x}_3) \tag{2}$$

where the following relative coordinates are used :

$$\begin{aligned} \underset{\sim}{x}_1 &= \frac{1}{\sqrt{2}}(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \\ \underset{\sim}{x}_2 &= \frac{1}{2}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 - \underset{\sim}{r}_3 - \underset{\sim}{r}_4) \\ \underset{\sim}{x}_3 &= \frac{1}{\sqrt{2}}(\underset{\sim}{r}_3 - \underset{\sim}{r}_4) \end{aligned} \tag{3}$$

which give again

$$\frac{1}{4}\sum_{i<j}^{4} r_{ij}^{2} = x_1^2 + x_2^2 + x_3^2 = X^2 \tag{4}$$

Defining $\underset{\sim}{y}_i$ and $\underset{\sim}{z}_i$ as in (3. 17) we have

$$\begin{aligned} r_3 &= \frac{1}{\sqrt{2}}\,\underset{\sim}{x}_3 - \frac{1}{2}\,\underset{\sim}{x}_2 \\ &= \frac{1}{2}\,\underset{\sim}{y}_2 - \frac{1}{\sqrt{2}}\,\underset{\sim}{y}_1 \\ &= \frac{1}{\sqrt{2}}\,\underset{\sim}{z}_1 + \frac{1}{2}\,\underset{\sim}{z}_2 \end{aligned} \tag{5}$$

The expressions for ψ_F (1234) are as follows :

Scalar Meson.

The only wave function is

$$\psi_F = K_{1234}\, Q(12)\, \chi(\underset{\sim}{x}_3) \tag{6}$$

which gives the matrix element

$$H(S) = -\lambda_1^2 \lambda_2^2 \,\frac{\pi}{R}\, \Phi_3 \tag{7}$$

where

$$\begin{aligned} \Phi_3 = \frac{(4\pi)^2}{\sqrt{2}\,k^4} \int_0^\infty \chi(x_3)\, x_3^2\, dx_3 \\ \times \int_0^\infty \phi\left(\sqrt{t^2 + x_3^2}\right) J_4(kt)\, t^5\, dt \end{aligned} \tag{8}$$

Pseudoscalar Meson.

There are two wave functions $\psi_F(1234)$:

(a) $$\frac{1}{\sqrt{3}} A_{12}\, \chi(\underset{\sim}{x}_3)\, u_0(1) \sum_{m=-1}^{1} (-)^m u_1^m(2)\, S_{34}^{-m} \qquad \text{(9a)}$$

(b) $$\frac{1}{\sqrt{6}}\, \chi(\underset{\sim}{x}_3)\, u_0(2) \begin{vmatrix} u_1^1(1) & u_1^0(1) & u_1^{-1}(1) \\ S_{12}^1 & S_{12}^0 & S_{12}^{-1} \\ S_{34}^1 & S_{34}^0 & S_{34}^{-1} \end{vmatrix} \qquad \text{(9b)}$$

which give the matrix elements

$$H_a(PS) = -\lambda_1 \lambda_2^2 \frac{\pi}{R} \Phi_4 \qquad \text{(10a)}$$

$$H_e(PS) = -\lambda_1^2 \lambda_2 \frac{\pi}{R} \Phi_4 \qquad \text{(10b)}$$

where

$$\Phi_4 = \frac{(4\pi)^2}{k^3} \int_0^\infty \chi(x_3)\, x_3^2\, dx_3 \int_0^\infty \phi(\sqrt{t^2+x_3^2})\, J_3(kt)\, t^4\, dt \qquad \text{(11)}$$

For the transition probabilities we find

$$P_d(S) = \frac{3\pi}{512} \cdot \frac{1}{E_2} \cdot \left(\frac{2ME_2}{k^2}\right)^5 |\Phi_3|^2 \qquad \text{(12a)}$$

$$P_d(PS) = \frac{\pi}{32} \cdot \frac{1}{E_2} \left(\frac{2ME_2}{\hbar^2} \right)^4 |\Phi_4|^2 \tag{12b}$$

The energy spectra of the deuterons are

$$dN_d(S) = (\epsilon_0 - \epsilon)^{3/2} \epsilon^{3/2} d\epsilon \tag{13a}$$

$$dN_d(PS) = (\epsilon_0 - \epsilon)^{1/2} \epsilon^{1/2} d\epsilon \tag{13b}$$

where $\epsilon_0 = \frac{\epsilon}{2}$ is the maximum energy of the deuterons , about 57 MeV. The deuterons in the scalar case have the'standard'spectrum. These energy spectra are shown in Figure 4.

4. 2. Triton Emission

The last reaction we consider is

$$He^4 + \pi^- \rightarrow H^3 + n \tag{14}$$

of which only the transition probability is required since the triton has a definite energy of about 30 MeV.

The triton wave function is supposed to be separable in the form

$$\psi_t^{\pm\frac{1}{2}}(123) = A_{12} \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} \xi(x) \tag{15}$$

where

$$x^2 = \frac{1}{3} \sum_{i<j}^{3} r_{ij}^2 = x_1^2 + x_2^2 \tag{16}$$

Particle 3 is taken as the proton and the relative coordinates are the same as for proton emission (3. 2).

The expressions for ψ_F (1234) are :

Scalar meson ,

$$\psi_F = A_{12} A_{34} \, u_0(3) \, \xi(x) \tag{17a}$$

Pseudoscalar meson ,

$$\psi_F = \frac{1}{\sqrt{3}} A_{12} \, \xi(x) \sum_{m=-1}^{1} (-)^m u_1^m(3) \, S_{34}^{-m} \tag{17b}$$

which give the matrix elements

$$H(S) = \sqrt{\frac{\pi k}{R}} \, \Phi_5 \tag{18a}$$

$$H(PS) = \sqrt{\frac{\pi k}{R}} \, \Phi_6 \tag{18b}$$

where

$$\Phi_5 = \frac{2\pi^{7/2}}{3} \int_0^\infty \xi(x)\, x^5\, dx$$

$$\times \int_0^\infty \phi(\sqrt{x^2+t^2})\, J_{1/2}(kt)\,(2t^2-x^2)\, t^{3/2}\, dt \qquad (19)$$

and

$$\Phi_6 = \frac{4\pi^{7/2}}{\sqrt{3}} \int_0^\infty \xi(x)\, x^5\, dx$$

$$\times \int_0^\infty \phi(\sqrt{x^2+t^2})\, J_{3/2}(kt)\, t^{5/2}\, dt \qquad (20)$$

The transition probabilities are

$$P_t(S) = \frac{M}{\hbar^2} |\Phi_5|^2 \qquad (21a)$$

$$P_t(PS) = \frac{M}{\hbar^2} |\Phi_6|^2 \qquad (21b)$$

4. 3. Relative probabilities of Different Reactions

In order to calculate the relative probabilities, it is necessary to particularize the spatial parts of the wave functions representing bound states, which up till now have only been restricted by a functional relationship of the form (2. 17). The wave functions chosen fall into two classes :

(A) Gaussian wave functions.

$$\phi(X) = e^{-aX^2} \qquad (He^4)$$

$$\chi(t) = \left(\frac{2\ell}{\pi}\right)^{3/4} e^{-\ell t^2} \qquad (H^2) \qquad (22)$$

$$\xi(x) = \left(\frac{2c}{\pi}\right)^{\frac{3}{2}} e^{-cx^2} \qquad (H^3)$$

where, as before,

$$X^2 = \frac{1}{4}\sum_{i<j}^{4} r_{ij}^2 \quad ; \qquad x^2 = \frac{1}{3}\sum_{i<j}^{3} r_{ij}^2$$

and a , b and c are parameters with dimensions length $^{-2}$. The integrals Φ are now easily found and give the following expressions for the transition probabilities :

$$P_p(s) = \frac{16\,\pi^4}{1155}\left(\frac{ME_1}{2\hbar^2 a}\right)^{\frac{13}{2}} \frac{1}{a^{\frac{13}{2}} E_1}\, e^{-\frac{ME_1}{\hbar^2 a}} \tag{23a}$$

$$P_d(s) = \frac{3\,\pi^{9/2}}{16\sqrt{2}}\left(\frac{ME_2}{2\hbar^2 a}\right)^5 \frac{(ab)^{3/2}}{(a+b)^3}\, \frac{1}{a^{13/2} E_2}\, e^{-\frac{ME_2}{\hbar^2 a}} \tag{23b}$$

$$P_t(s) = \frac{16\,\pi^4}{9}\left(\frac{ME_3}{2\hbar^2 a}\right)^{3/2} \left(\frac{3c}{a+c} - \frac{ME_3}{\hbar^2 a}\right)^2$$

$$\times\, \frac{(ac)^3}{(a+c)^6}\, \frac{1}{a^{13/2} E_3}\, e^{-\frac{ME_3}{\hbar^2 a}} \tag{23c}$$

$$P_p(Ps) = \frac{16\,\pi^4}{105}\left(\frac{ME_1}{2\hbar^2 a}\right)^{\frac{11}{2}} \frac{1}{a^{11/2} E_1}\, e^{-\frac{ME_1}{\hbar^2 a}} \tag{24a}$$

$$P_d(PS) = \sqrt{2}\,\pi^{9/2}\left(\frac{ME_2}{2\hbar^2 a}\right)^4 \frac{(ab)^{3/2}}{(a+b)^3}\frac{1}{a^{11/2}E_2}\, e^{-\frac{ME_2}{\hbar^2 a}} \tag{24b}$$

$$P_t(PS) = \frac{64\pi^4}{3}\left(\frac{ME_3}{2\hbar^2 a}\right)^{5/2}\frac{(ac)^3}{(a+c)^6}$$

$$\times \frac{1}{a^{11/2}E_3}\, e^{-\frac{ME_3}{\hbar^2 a}} \tag{24c}$$

The choice of parameters a and c has been discussed by Bruno (1948) , who put

$$c = \frac{3}{4}a \; ; \quad \frac{1}{\sqrt{a}} = 2 \cdot 10^{-13} \text{ cm} \tag{25}$$

We adopt these values and also put $b = \frac{a}{2}$ to get the relative probabilities shown in Table 1.

Table 1.

	proton	deuteron	triton
S	71	7	1
PS	223	22	1

(B) Wave functions of "asymptotic" type :

$$\phi(X) = \frac{1}{X} e^{-\alpha X}$$

$$\chi(t) = \sqrt{\frac{\beta}{2\pi}} \frac{1}{t} e^{-\beta t} \qquad (26)$$

$$\xi(x) = \sqrt{\frac{8}{3\pi^3}} \frac{\gamma^2}{x} e^{-\gamma x}$$

where α , β and γ are parameters with the dimensions of length $^{-1}$.

The wave functions (26) give the following expressions for the transition probabilities :

$$P_p(S) = \frac{2^{21} . 15 . \pi^3}{77 . \alpha^{11} E_1} \left(\frac{2ME_1}{\hbar^2 \alpha^2}\right)^{\frac{13}{2}} \frac{1}{\left(1 + \frac{2ME_1}{\hbar^2\alpha^2}\right)^{12}} \qquad (27a)$$

$$P_d(S) = \frac{675 . \pi^4}{8 E_2} \frac{\alpha^{10}\beta}{\zeta_2^{18}(\zeta_2+\beta)^{12}} \left(\frac{2ME_2}{\hbar^2\alpha^2}\right)^5$$

$$\times\left(63\zeta_2^4 + 122\beta\zeta_2^3 + 102\beta^2\zeta_2^2 + 42\beta^3\zeta_2 + 7\beta^4\right) \qquad (27b)$$

$$P_t(S) = \frac{135\,\pi^5 \gamma^4 \alpha^3}{4 E_3 \zeta_3^3 (\zeta_3 + \gamma)^{15}} \left(\frac{2ME_3}{\hbar^2 \alpha^2}\right)^{\frac{3}{2}} A^2 \tag{27c}$$

where

$$A = \frac{\alpha^2}{\zeta_3^4}\left(504\zeta_3^2 + 260\,\gamma\zeta_3 + 42\,\gamma^2\right)$$

$$- \frac{1}{\zeta_3^2}\left(485\zeta_3^2 + 68\,\gamma\zeta_3 + 12\,\gamma^2\right)$$

$$P_p(PS) = \frac{3 .\, 2^{19} \pi^3}{35 .\, \alpha^9 .\, E_1}\left(\frac{2ME_1}{\hbar^2\alpha^2}\right)^{\frac{11}{2}} \frac{1}{\left(1 + \frac{2ME_1}{\hbar^2\alpha^2}\right)^{10}} \tag{28a}$$

$$P_d(PS) = \frac{36\,\pi^4 \beta}{E_2 (\zeta_2 + \beta)^{10}} \cdot \frac{1}{\left(1 + \frac{\hbar^2\alpha^2}{2ME_2}\right)^4} \tag{28b}$$

$$\times \left(35 + 47\frac{\beta}{\zeta_2} + 25\left(\frac{\beta}{\zeta_2}\right)^2 + 5\left(\frac{\beta}{\zeta_2}\right)^3\right)^2$$

$$P_t(PS) = \frac{35^2 .\, 99^2 .\, \pi^5 \gamma^4}{64 .\, E_3 (\zeta_3 + \gamma)^{13}} \cdot \frac{1}{\left(1 + \frac{\hbar^2\alpha^2}{2ME_3}\right)^{5/2}} \tag{28c}$$

In these expressions we have put

$$\zeta_i^2 = \alpha^2 + \frac{2ME_i}{\hbar^2}$$

Putting $\alpha^2 = \frac{2MB_4}{\hbar^2}$ where B_4 is the binding energy of He^4, and similarly defining β and γ in terms of the binding energies of deutrron and triton respectively, we find the results shown in Table 2.

Table 2.

Relative Probabilities from

"asymptotic" wave functions

	proton	deuteron	triton
S	30	4.5	1
PS	79	18	1

4. 4 The Lifetime for Capture

In the previous sections we have shown how the relative probabilities of different slow meson reactions in He^4 may be estimated. We now briefly discuss the order of magnitude of the absolute transition probability , which determines the lifetime of the mesic-nuclear atom .

Considering the hamiltonian equations for the nucleons , we take the following expressions for the interaction energy with the meson field :

$$H' = f\, mc^2 \left(\frac{\hbar}{mc}\right)^{\frac{3}{2}} \psi\, \Gamma + \text{c.c.} \qquad \text{(scalar)} \quad (29)$$

$$H' = g\, mc^2 \left(\frac{\hbar}{mc}\right)^{5/2} \underset{m}{\sigma} \cdot \underset{m}{\nabla} \psi\, \Gamma + \text{c.c.} \qquad \text{(pseudoscalar)} \quad (30)$$

where m is the meson mass , ψ is the meson wave function ,

$$\psi = \frac{\gamma^{3/2}}{\sqrt{\pi}}\, e^{-\gamma r}$$

where $\gamma = 2me^2/\hbar^2$, and f and g are dimensionless coupling constants.

According to § 2. 1 , equation (9) et seq. , we retain only the third term in the series expansion of the exponential function , obtaining

$$H' = \frac{2^{5/2}}{(137)^{7/2}\sqrt{\pi}} \, f \, mc^2 \left(\frac{mc}{\hbar}\right)^2 r^2 \qquad \text{(scalar)} \quad (31)$$

$$H' = \frac{2^{7/2}}{(137)^{7/2}\sqrt{\pi}} \, g \, mc^2 \, \frac{mc}{\hbar} \, \underset{\sim}{\sigma} \cdot \underset{\sim}{r} \qquad \text{(pseudoscalar)} \quad (32)$$

The transition probability is

$$P = \frac{2\pi}{\hbar} \int \sum |H|^2 \, d\rho_E$$

Using the values of $W = \sum |H|^2$ previously calculated for proton emission , we find the following values for P :

$$P(S) = 1.15 \, f^2 \, 10^9 \ \text{sec}^{-1} \qquad \text{(scalar)} \quad (33)$$

$$P(PS) = 1.2 \, g^2 \, 10^9 \ \text{sec}^{-1} \qquad \text{(pseudoscalar)} \quad (34)$$

In order to estimate the magnitudes of f and g , we have recourse to the usual perturbation theory of nuclear forces which gives , in the scalar case , the potential

$$V(r) = f^2 \frac{mc^2}{2\pi} \frac{e^{-\lambda r}}{\lambda r}$$

where $\lambda = \frac{mc}{\hbar}$. Putting $f^2 \simeq \pi$, we have

$f^2 \frac{mc^2}{2\pi} \simeq$ 70 MeV., which is of the right order of magnitude. Similarly we have $g^2 \simeq \pi$.

We now find the lifetimes for capture, which are equal to $1/P$, to be about $3. 10^{-10}$ secs.for both scalar and pseudoscalar mesons. This is a good deal less than the lifetime for decay of the meson ,and shows that the meson will in fact be absorbed by the He^4 nucleus , causing a disintegration.

4. 5. Summary and Discussion on Meson Capture by He^4

It has been shown that by making certain assumptions the theory of π^- -meson capture by He^4 may be developed to yield expressions for such observable quantities as the energy spectra of the emitted particles and the relative probabilities of different reactions. The main assumptions made were that

(i) the meson is initially in a Bohr orbit around the He^4 nucleus ; thus the capture in flight of a fast meson is excluded ;

(ii) the nucleons are always so slow that the non-relativistic expressions for the meson-nucleon interaction may be used ;

(iii) first order perturbation is valid.

The second of these assumptions is open to question since the nucleons can have energies up to 90 MeV., but it is not unreasonable to make this approximation in a simple theory of the kind here attempted. The same may be said of the third assumption , that it is the best we can do in the present uncertain state of meson theories.

Other assumptions made concerned the form of the wave functions. The final-state wave functions were free waves , chosen so that the parity and angular momentum properties were directly exhibited. Wave functions for bound states were assumed to be arbitrary functions $\phi(x)$, with $x^2 \propto \sum r_{ij}^2$. It is noteworthy that at this stage , without choosing the functional form ϕ , all energy spectra and angular distributions of the emitted particles may be found.

Referring to Figure 1, it will be seen that the proton spectra in the scalar and pseudoscalar cases are nearly the same , although they differ somewhat from the "standard". This difference can be explained as an effect of the exclusion principle , as follows. For the proton to have its maximum energy of 84 MeV, the three neutrons must each have the same momentum , which is forbidden by the Pauli principle. Thus the scalar and pseudoscalar spectra vanish more strongly at the high energy end.

As shown by Figure 2, the neutron spectra in all three cases are nearly the same . The other case shown here was calculated with scalar interaction and plane waves as final state wave functions . Since it is so different from the other cases , we see that angular momentum and parity considerations can have a marked effect even in such a complicated case as this . In general , however , this effect is small , and the spectra obtained from plane wave functions are always the same in the pseudoscalar case. The scalar proton spectrum is also the same.

The angular distribution of the neutrons is also insensitive to the interaction , as shown by Figure 3. The deuteron energy spectra , however , (Figure 4) show a considerable dependence on the meson interaction. This is no doubt a consequence of the presence of just two neutrons in the final state , as in the case of meson capture by deuterium. Since the final state only contains three particles , there is a considerable reduction in the number of permitted final states over the case of proton emission.

The possibility of distinguishing between different meson interactions by measurement of the deuteron spectrum is not great because deuteron emission is much rarer than proton emission. Tables 1 and 2 show that the relative probabilities of different reactions do not depend much on the choice of wave functions for bound states , but the differences from this cause are sufficient to mask the differences due to the meson interaction. In all cases proton emission is the most , and triton emission the least , probable. This seems to be due to the fact that the number of permitted final states increases very rapidly with the number of emitted particles , provided the kinetic energy available is high enough. If the kinetic energy derived from the meson's rest energy were very small , as would be the case if a photon were emitted in the capture process , then presumably triton emission would be the commonest reaction.

5 The capture of π^- mesons by C^{12}

5. 1. General Considerations

The work of the previous sections shows that very little would be gained by an experimental study of meson capture by He^4. Also there is not much hope of such difficult experiments being made. The question now arises whether the calculations can be made to apply to some other nucleus more amenable to observation. We choose C^{12} for two reasons. Firstly it is particularly simple, being an α-particle nucleus, and so should allow an α-particle model to be used. In the second place some observations have already been made on meson reactions in C^{12}, notably by Menon, Muirhead and Rochat (referred to as M.M.R.). They showed that in a considerable proportion of cases π^- capture by C^{12} leads to emission of two α-particles and one singly-charged particle, together with neutrons. We see that we can describe this case by an α-particle model of C^{12} provided we let only one of the α-particles disintegrate. This means that the α-particles in C^{12} are considered to have a real existence and that one of tham captures the meson. This α-particle then disintegrates in one of the ways previously described. The remaining α-particles will take up the recoil and will be emitted either directly or in the form of Be^8. Study of the mutual angular correlation of the two α-particles will throw light on this last point, since the Be^8, possibly in an excited state, will generally have fairly high kinetic energy, thus causing the disintegration α-particles to have nearly the same direction of motion.

Since proton emission is the most probable event in the disintegration of He^4, we expect the same to be true for C^{12}. Also the type of meson theory assumed has only a slight effect on the proton spectra from He^4; its effect on the α-particle energy spectrum from C^{12} should be negligible. Accordingly we consider only one reaction

$$C^{12} + \pi^- \rightarrow 2He^4 + H^1 + 3n \qquad (1)$$

and only one meson interaction, say pseudoscalar. The calculations are quite similar to the preceding ones and can be described more briefly.

5. 2. Waves Functions and Matrix Elements

In the initial state particles 1 and 2 are neutrons , 3 and 4 protons and 5 and 6 α-particles. The interaction with the meson is taken as proportional to $\underset{\sim}{\sigma}_3 \cdot \underset{\sim}{\Gamma}_3$ and turns 3 into a neutron.

The relative coordinates are as follows :

$$\begin{aligned}
\underset{\sim}{x}_1 &= \frac{1}{\sqrt{2}}\left(\underset{\sim}{r}_1 - \underset{\sim}{r}_2\right) \\
\underset{\sim}{x}_2 &= \sqrt{\frac{2}{3}}\left(\frac{1}{2}\left(\underset{\sim}{r}_1 + \underset{\sim}{r}_2\right) - \underset{\sim}{r}_3\right) \\
\underset{\sim}{x}_3 &= \frac{\sqrt{3}}{2}\left(\frac{1}{3}\left(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3\right) - \underset{\sim}{r}_4\right) \\
\underset{\sim}{x}_4 &= \sqrt{2}\left(\frac{1}{4}\left(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4\right) - \underset{\sim}{r}_5\right) \\
\underset{\sim}{x}_5 &= \sqrt{\frac{2}{3}}\left(\frac{1}{4}\left(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4\right) + \underset{\sim}{r}_5 - 2\underset{\sim}{r}_6\right)
\end{aligned} \tag{2}$$

The spatial wave function of the α-particle which will disintegrate into separate nucleons 1 , 2 , 3 and 4 is again taken as $\phi(X)$, with

$$X^2 = \frac{1}{4}\sum_{i<j}^{4} r_{ij}^2 = x_1^2 + x_2^2 + x_3^2 \tag{3}$$

The wave function describing the initial C^{12} nucleus is

$$A_{12}\, A_{34}\, \phi(X)\, F(Y) \tag{4}$$

where

$$Y^2 = \frac{4}{3} \sum_{i<j=5}^{7} r_{ij}^2 = x_4^2 + x_5^2 \tag{5}$$

putting

$$\mathbf{r}_7 = \frac{1}{4}\left(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 + \mathbf{r}_4\right) \tag{6}$$

The forms of the functions ϕ , F are temporarily left open. The internal wave functions of the α-particles 5 and 6 are not required and are omitted from (4).

The final state wave functions are again taken as free waves and in this case are just the expressions (3. 16) multiplied by $U_0(4)\, U_0(5)$. The corresponding matrix elements similarly can be derived from (3. 26) by multiplying by

$$\int F(Y)\, U_0(4)\, U_0(5)\, d\mathbf{x}_4\, d\mathbf{x}_5 \tag{7}$$

We easily find that

$$W = \sum |H|^2$$

$$\propto \frac{\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2 \lambda_5^2 (\lambda_1^2 + \lambda_2^2)}{\lambda^9 \; (k^2 - \lambda^2)} \, |\Phi \, G|^2 \qquad (8)$$

where

$$\Phi(\lambda) = \int_0^\infty \phi(u) \, J_{9/2}(\lambda u) \, u^{11/2} \, du \qquad (9)$$

$$G(\lambda) = \int_0^\infty F(t) \, J_2\left(t\sqrt{k^2 - \lambda^2}\right) t^3 \, dt \qquad (10)$$

and $$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \qquad (11)$$

It will be seen that the situation is more complicated than the previous case , for now the integrals Φ and G depend not only on the constant k , but also on λ . Roughly speaking , the value of λ determines the way in which the kinetic energy is shared between the free nucleons on the one hand and the two α-particles on the other. Thus the energy spectra of the emitted particles will now depend on the form of the initial wave functions . However , the energy spectrum of the α-particles has not been measured very accurately by M. M. R., and the experimental spectrum can be explained on the basis of either Gaussian or "asymptotic" wave functions.

Choosing Gaussian wave functions , we put

$$\left.\begin{aligned} \phi(u) &= e^{-au^2} \qquad (He^4) \\ F(t) &= e^{-\mu t^2} \qquad (C^{12}) \end{aligned}\right\} \tag{12}$$

and easily find

$$\Phi = \frac{\lambda^{9/2}}{(2a)^{11/2}} e^{-\frac{\lambda^2}{4a}} \tag{13}$$

$$G = \frac{k^2-\lambda^2}{(2\mu)^3} e^{-(k^2-\lambda^2)/4\mu} \tag{14}$$

$$W \propto \lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2 \lambda_5^2 (\lambda_1^2+\lambda_2^2) e^{-\left(\frac{1}{a}-\frac{1}{\mu}\right)\frac{\lambda^2}{2}} \tag{15}$$

retaining in the expression for W only those factors which will affect the energy spectra.

5. 3. Energy Spectra of the Emitted α-particles

The final state wave functions have been chosen to represent states in which particle 6 has a definite energy $\epsilon = \hbar^2 \lambda_5^2 / 3M$ where M is the nucleon mass.
From

$$\int W \, d\rho_E \propto \frac{d}{dE} \int W \, d\lambda_1 \, d\lambda_2 \, d\lambda_3 \, d\lambda_4 \, d\lambda_5$$

we find the α-particle energy spectrum :

$$\frac{dN_\alpha}{d\epsilon} = \lambda_5^2 \frac{d\lambda_5}{d\epsilon} \cdot \frac{d}{dE} \int \lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_4^2 (\lambda_1^2 + \lambda_2^2)$$

$$\times e^{\gamma \lambda^2 / k^2} \, d\lambda_1 \, d\lambda_2 \, d\lambda_3 \, d\lambda_4 \tag{16}$$

the region of integration being

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \equiv \lambda^2 < k^2 - \lambda_4^2 - \lambda_5^2 \tag{17}$$

In (16) we have put

$$\frac{1}{\mu} - \frac{1}{a} = \frac{2\gamma}{k^2} \; ; \tag{18}$$

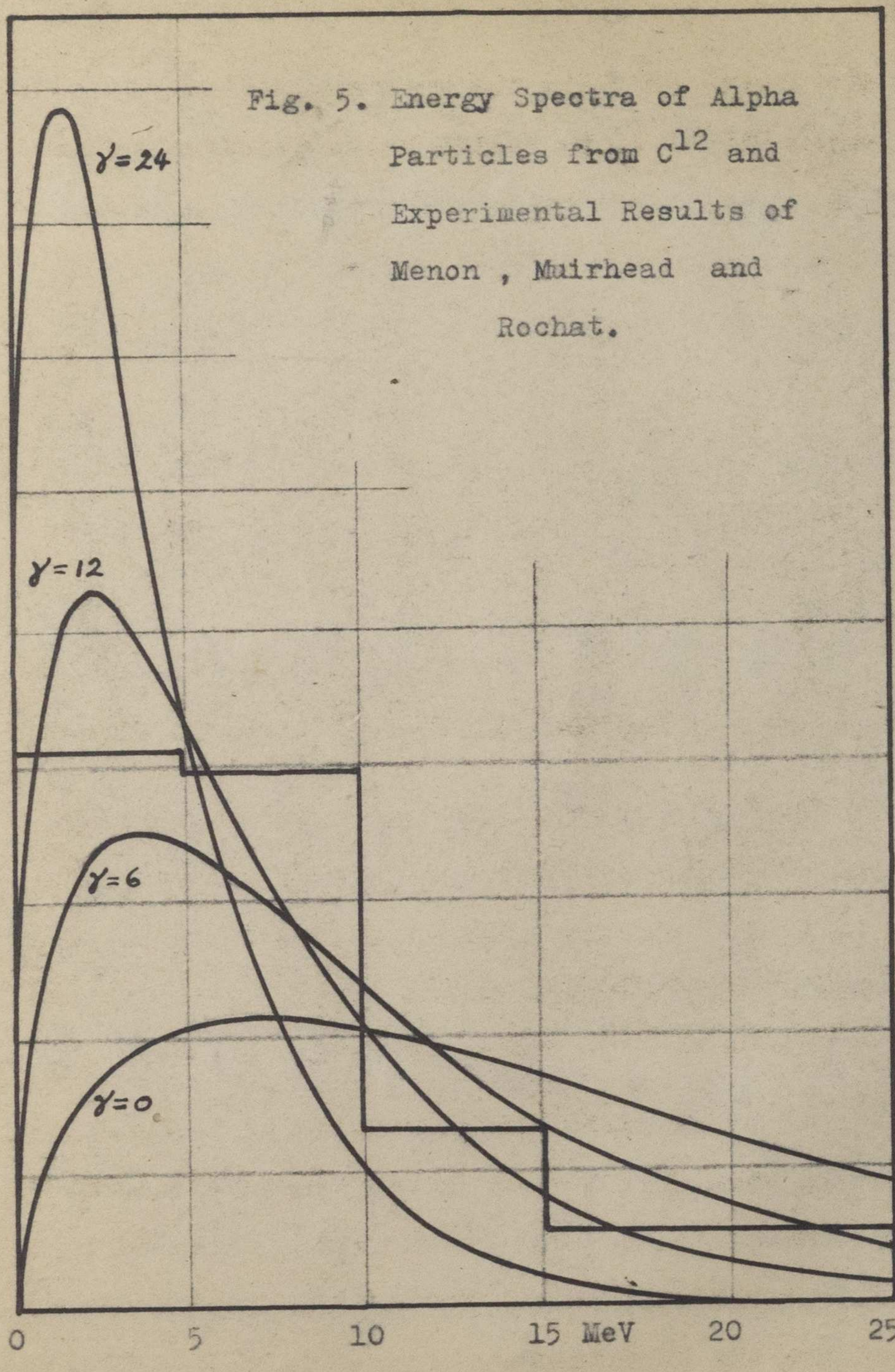


Fig. 5. Energy Spectra of Alpha Particles from C^{12} and Experimental Results of Menon , Muirhead and Rochat.

this definition of γ is the same as in Clark and Ruddlesden (1951) . By taking polar coordinates in the $\lambda_1 \lambda_2 \lambda_3$ space and integrating over λ_4 , (16) gives

$$\frac{dN_\alpha}{d\epsilon} = \epsilon^{1/2} \int e^{\gamma \lambda^2 / k^2} \left(k^2 - \lambda_5^2 - \lambda^2\right)^{1/2} \lambda^{10} \, d\lambda \tag{19}$$

The substitution $\lambda = \left(k^2 - \lambda_5^2\right)^{1/2} t^{1/2}$ now gives

$$\frac{dN_\alpha}{d\epsilon} = (\epsilon_0 - \epsilon)^6 \, \epsilon^{1/2} \int_0^1 e^{\gamma \left(1 - \frac{\epsilon}{\epsilon_0}\right) t} (1-t)^{1/2} \, t^{9/2} \, dt \tag{20}$$

where ϵ_0 is the maximum value of ϵ , about 70 MeV.

The integral in (20) is a hypergeometric function of Kummer's type and cannot be expressed in terms of elementary functions . It has been calculated numerically for different values of γ , the corresponding values of $\frac{dN_\alpha}{d\epsilon}$ being shown in Figure 5 , together with the experimental results of M. M. R. It will be seen that $\gamma = 12$ gives reasonable agreement with experiment. The case $\gamma = 0$, which is in effect just the statistical factor , is inadequate to describe the experiments.

5. 4. The Nuclear Radius of C^{12}

We have seen that the energy spectrum of the emitted α -particles depends rather critically on the value assigned to the parameter γ , which fact makes it desirable to have an independent method of calculating γ . This can be done very simply since the radius of the nucleus represented by the wave functions (12) depends on γ .

Using (3) , the relative density of nucleons at distance $\underset{\sim}{r}$ from the centre of an α -particle is $\rho_1(\underset{\sim}{r})$ where

$$\rho_1(\underset{\sim}{r_1}) = \int |\phi(x)|^2 \, d\underset{\sim}{r_2} \, d\underset{\sim}{r_3}$$

$$\propto e^{-\frac{8}{3} a r^2} \tag{21}$$

Similarly the relative density of α -particles in the C^{12} nucleus is

$$\rho_2(\underset{\sim}{r_5}) = \int |F(Y)|^2 \, d\underset{\sim}{r_6}$$

$$\propto e^{-2\mu r_5^2} \tag{22}$$

The relative density of nucleons at distance $\underset{\sim}{r}$ from the centre of C^{12} is now

$$\rho_c(\underset{\sim}{r}) = \int \rho_2(\underset{\sim}{r}') \, \rho_1(\underset{\sim}{r} - \underset{\sim}{r}') \, d\underset{\sim}{r}' \propto e^{-\sigma r^2} \tag{23}$$

where

$$\sigma = \frac{24 a\mu}{9\mu + 2a}$$

From this distribution we find the mean value of r^2 :

$$\overline{r^2} = \frac{3}{2\sigma} \tag{24}$$

Consider now a simplified model of the C^{12} nucleus with density distribution given by

$$\rho_0(r) = \begin{cases} 1, & r < R_c \\ 0, & r > R_c \end{cases} \tag{25}$$

This distribution gives

$$\overline{r^2} = \frac{3}{5} R_c^2 \tag{26}$$

Comparing (24) and (26) we get

$$R_0^2 \sigma = \frac{5}{2} \tag{27}$$

Giving R_c the value obtained from Gamow's formula for the radius of C^{12}, 2.79×10^{-13} cm, and taking $\frac{1}{\sqrt{a}} = 2 \times 10^{-13}$ cm as before, with $\frac{k^2}{2a} = 10.3$ we find $\gamma = 40$. However, C^{12} being an exceptionally stable nucleus, we might expect its radius to be less than the value given by Gamow's formula. Taking $R_c = 2.5 \times 10^{-13}$ cm we find $\gamma = 21$. Although an improvement, this value of γ is still unacceptable. If $1/\sqrt{a}$ is increased to 2.15×10^{-13} cm, $R_c = 2.5 \times 10^{-13}$ cm gives $\gamma = 12$. If R_α, the radius of He^4, is defined in the corresponding way to R_c, this value of a gives $R_\alpha = 2.08 \times 10^{-13}$ cm, compared with Gamow's value $1{\cdot}94 \times 10^{-13}$ cm.

Thus agreement with experiment can only be achieved by reducing the radius of C^{12} and increasing that of He^4 from their best theoretical values. The differences are rather small, and the theory may be considered satisfactory since Gaussian wave functions cannot be expected to represent the nuclear radius very exactly.

5. 5. Indirect Disintegration of C^{12}

Observations on nuclear explosions in light nuclei have shown that quite often two α-particles are emitted with nearly the same energies and directions of motion. It was conjectured that they were emitted as a Be^8 nucleus with a very short lifetime. In the case of π^- capture by C^{12} this could well be the most probable reaction , since Be^8 is the residual nucleus after a single α-particle has been removed.

We consider the reaction

$$\left.\begin{array}{l} C^{12} + \pi^- \rightarrow Be^8 + H^1 + 3n \\ \quad Be^8 \rightarrow 2\,He^4 \end{array}\right\} \tag{28}$$

in which the Be^8 nucleus may be in an excited state. Indeed , Telegdi and Zünti (1950) found that they could explain the energy spectrum of the α-particles emitted in the photo-disintegration of C^{12} by assuming that the only state of Be^8 involved was the excited state at 3 MeV.

The appropriate relative coordinates are $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ and $\underset{\sim}{x}_3$ given by (2), together with

$$\underset{\sim}{\xi}_4 = \sqrt{2}\left(\tfrac{1}{2}(\underset{\sim}{r}_1 + \underset{\sim}{r}_2 + \underset{\sim}{r}_3 + \underset{\sim}{r}_4) - \underset{\sim}{r}_5 - \underset{\sim}{r}_6\right) \tag{29}$$

$$\underset{\sim}{\xi}_5 = \sqrt{6}\,(\underset{\sim}{r}_5 - \underset{\sim}{r}_6)$$

The final state wave functions are again derived from (3. 16), this time being multiplied by $u_0(\xi_4)\,\chi(\xi_5)$ where $\chi(\xi_5)$ is the internal wave function of the Be^8 nucleus.

With the Gaussian wave functions (12) we easily find

$$W \propto \lambda_1^2 \lambda_2^2 \lambda_3^2 (\lambda_1^2 + \lambda_2^2)\, N^2 \times e^{-\frac{\lambda^2}{2a} - \frac{3\lambda_4^2}{2\mu}} \tag{30}$$

where

$$N = \int e^{-\frac{1}{3}\mu r^2}\, \chi(\underset{\sim}{r})\, d\underset{\sim}{r} \tag{31}$$

It is obvious that the probability of this process is zero unless the spin of Be^{8*} is 0. This is a consequence of our assumption of (4) as a suitable wave function for C^{12}, since this represents a state in which all the three α-particles are in S states. In fact, the spin of the first excited state of Be^{8} is not known, although scattering experiments (reviewed by Hornyak et. al. 1950) seem to indicate that it is 1. However, this value seems very implausible, since the state in question is known to be very broad, indicating a very short lifetime against α-decay. α-decay, of course, is forbidden by the parity selection rule to a state with spin 1.

If the spin of this state is different from zero, we can only obtain a large probability for reaction (28) by assuming that the initial C^{12} wave function has a large admixture of states with higher angular momentum. However, assuming the spin of Be^{8*} to be zero, and representing it as a closed state by taking $\chi(\xi) = e^{-\rho\xi^2}$ with $\rho \simeq \frac{a}{3}$, we find that reaction (28) has about 20% probability compared with (1). We tentatively suppose that the emission probability of Be^{8*} with non-zero spin will be of the same order of magnitude as this. It is of course impossible to represent a state of Be^8 with odd angular momentum by means of an α-particle model.

The energy distribution of the emitted Be^{8*} does not depend on the wave function χ and is found to be

$$\frac{dN}{d\epsilon} = \epsilon^{\frac{1}{2}} (\epsilon_0 - \epsilon)^{9/2} e^{-\gamma' \frac{\epsilon}{\epsilon_0}} \tag{32}$$

with

$$\gamma' = \frac{3a - \mu}{a - \mu} \gamma \simeq 48 \tag{33}$$

The maximum energy of the Be^{8*} is $\epsilon_0 \simeq 35$ MeV.

In the centre of mass system of the Be^{8*}, each of the α-particles arising from its decay has energy U/2 , where U is the excitation energy of Be^{8*}, here assumed to be 3 MeV. The observed velocity of the α-particles is the resultant of the velocity of the Be^{8*} and the velocity of the

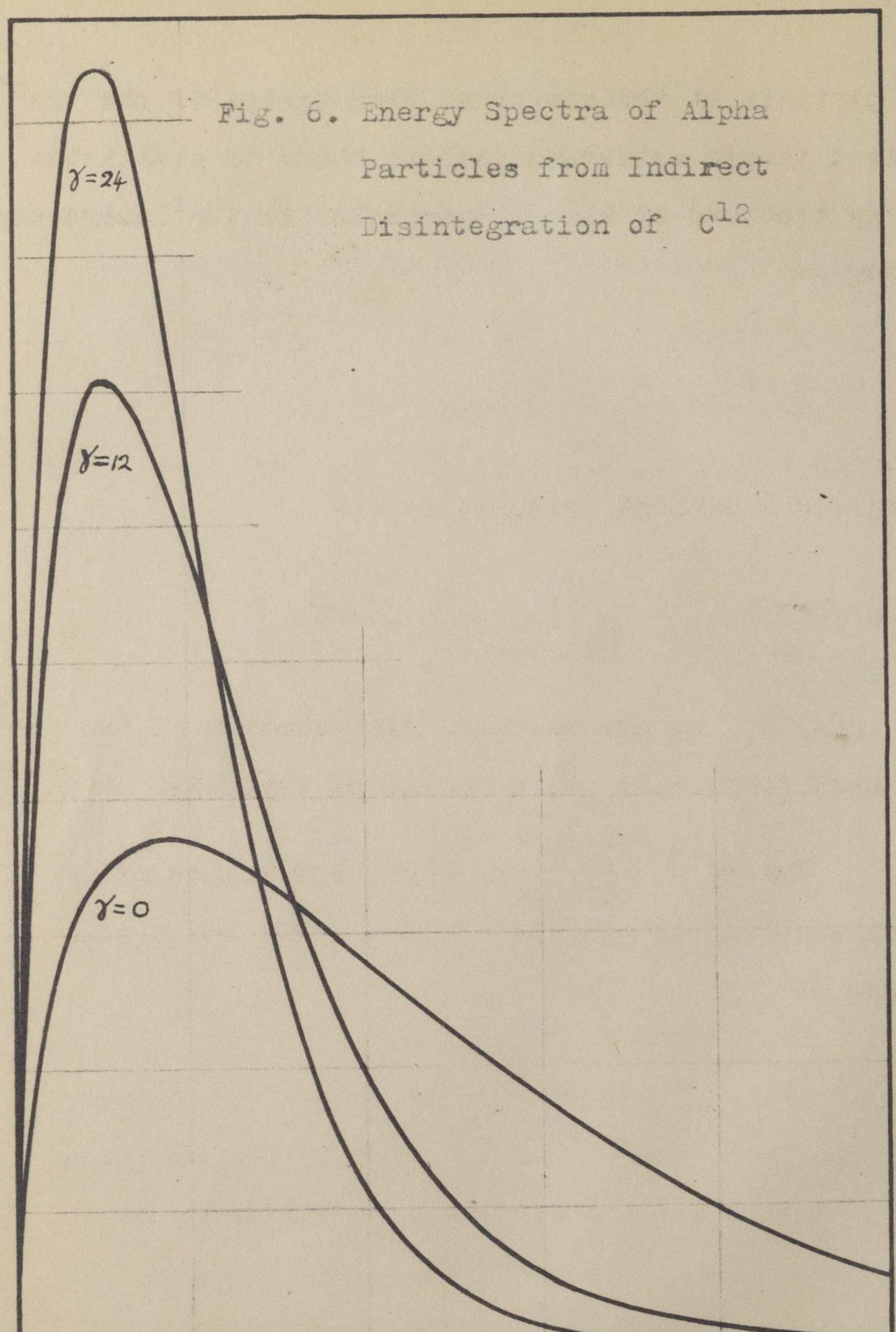


Fig. 6. Energy Spectra of Alpha Particles from Indirect Disintegration of C^{12}

α-particle in the centre of mass system of the Be^{8*}. It is a simple matter to combine these to obtain the energy spectrum of the α-particles in the laboratory system as

$$P(\epsilon_\alpha)\, d\epsilon_\alpha = d\epsilon_\alpha \int g(v)\, \frac{dv}{v} \tag{34}$$

where the limits of integration are

$$v = \frac{1}{\sqrt{2M}} \left| \sqrt{\frac{u}{2}} \pm \sqrt{\epsilon_\alpha} \right| \tag{35}$$

and g(v) dv is the velocity distribution of the Be^{8*}, obtained from (32) by a change of variable.

Curves $P(\epsilon_\alpha)$ for different values of γ, obtained by numerical integration, are shown in Figure 6.

5. 6. Angular Correlation of α -particles from C^{12}

When two or more charged particles are emitted in a nuclear disintegration , it frequently happens that their mutual angular correlation yields information about the state of the parent nucleus. The usual case is when the particles are emitted successively, for instance in an evaporation process , and then the observed angular distributions depend on the angular momenta of tha successive states. When the particles are emitted simultaneously from a light nucleus , the most important factor governing the angular correlation is the necessity to conserve momentum. This forces the particles to have a certain characteristic angular correlation even when they are emitted in S states. (In the evaporation case , of course , particles in S states have isotropic angular distributions.) This point is illustrated by the angular distributions of the neutrons from He^4 , which are almost independent of the angular momenta involved.

In the case of the α -particles emitted from C^{12} , whose mutual angular correlation will now be considered , the most important use of the measured angular correlation is to distinguish between the cases in which the α -particles are emitted in the form of Be^8 and those in which they are not. It is possible , however , that the angular correlation may depend on the angular momenta of the α -particles. According to the theory as developed in § 5. 2 , the α -particles are restricted to S states. This is purely a consequence of the initial wave function assumed , which we have seen is inadequate to explain the emission of Be^{8*} with non-zero spin. It turns out that the angular correlations of the α -particles in higher angular momentum states can be calculated without reference to the initial wave function.

The angular correlation of the α -particles from Be^{8*} is easily found since it only depends on $g(v)$, the velocity distribution of the Be^{8*} , and U , its excitation energy. Simple geometrical considerations give

$$N(\mu) = -\frac{1}{\mu^3}\int_{v_0 \cot\frac{\theta}{2}}^{v_0} \frac{(v_0^2 - v^2)^2}{\sqrt{4v_0^2v^2 - (v_0^2-v^2)^2\tan^2\theta}} \cdot \frac{g(v)\,dv}{v}$$

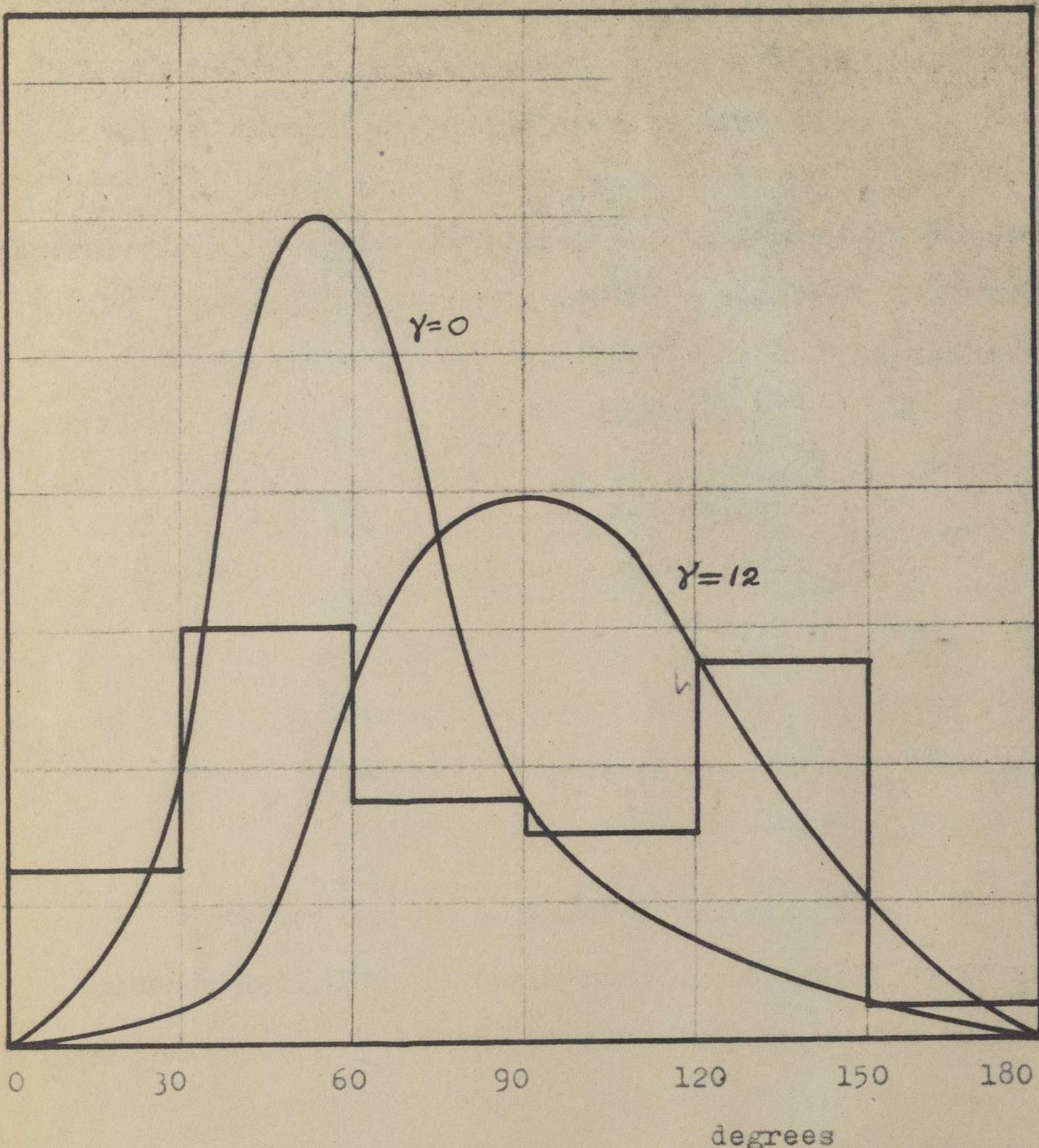


Fig. 7. Angular Correlations of Alpha Particles from Indirect Disintegration of C^{12} together with Experimental Results of Menon et al.

for the case $\frac{\pi}{2} < \theta < \pi$, where v_0^2 = U/4M and $\mu = \cos\theta$. There is an analogous expression for the case $0 < \theta < \frac{\pi}{2}$. The curves shown in Figure 7 have been calculated numerically from this expression for different values of γ . The experimental results of M. M. R.* are also shown in Figure 7.

Next consider the case which is permitted by the wave function (4) . Exactly as in § 3. 4 we write the final state wave functions in momentum space :

$$\Psi_F \propto \frac{1}{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} \int e^{-i\sum_1^5 \underset{\sim}{x}_i \cdot \underset{\sim}{k}_i} G(\lambda_i ; \underset{\sim}{k}_i)\, dk \qquad (36)$$

where $G(\lambda_i ; \underset{\sim}{k}_i) = \prod_1^5 \delta(\lambda_j - |\underset{\sim}{k}_j|)$ and dk = $d\underset{\sim}{k}_1\, d\underset{\sim}{k}_2\, d\underset{\sim}{k}_3\, d\underset{\sim}{k}_4\, d\underset{\sim}{k}_5$. The infinite normalization constant can be removed to give the angular correlation function

$$H(\mu)\, d\mu = \frac{1}{\lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2\lambda_5^2} \int^1 G(\lambda_i ; \underset{\sim}{k}_i)\, dk \qquad (37)$$

*Private communication.

where θ is the angle between the directions of emission of the α-particles, $\mu = \cos\theta$ and the prime on the integral sign means that the integration over μ is not to be carried out. We put

$$\sum_{1}^{5} \underset{\sim}{x}_i \cdot \underset{\sim}{k}_i = \sum_{1}^{6} \underset{\sim}{p}_i \cdot \underset{\sim}{r}_i$$

where the $\hbar \underset{\sim}{p}_i$ are the actual momenta of the particles. Solving for $\underset{\sim}{p}_i$ we obtain the particular cases

$$-\sqrt{2}\, \underset{\sim}{k}_4 = \underset{\sim}{p}_5 + \tfrac{1}{2} \underset{\sim}{p}_6$$

$$-2\sqrt{\tfrac{2}{3}}\, \underset{\sim}{k}_5 = \underset{\sim}{p}_6 \tag{38}$$

Now in (37) take $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$, $\underset{\sim}{k}_3$, $\underset{\sim}{p}_5$ and $\underset{\sim}{p}_6$ as independent variables. The integrals over $\underset{\sim}{k}_1$, $\underset{\sim}{k}_2$ and $\underset{\sim}{k}_3$ can be performed, giving constant multiplying factors which can be dropped, leaving

$$H(\mu) = \frac{1}{\lambda_4^2 \lambda_5^2} \int \delta\left(\sqrt{2}\,\lambda_4 - |\underset{\sim}{p}_5 + \tfrac{1}{2}\underset{\sim}{p}_6|\right)$$

$$\times\, \delta\left(2\sqrt{\tfrac{2}{3}}\,\lambda_5 - |\underset{\sim}{p}_6|\right) p_5^2\, p_6^2\, dp_5\, dp_6 \tag{39}$$

This expression is the angular correlation of the α-particles corresponding to a single wave function. To obtain the observed angular correlation P we must integrate over the final states :

$$P(\mu) = \frac{d}{dE} \int W . H(\mu) \, d\lambda_1 \, d\lambda_2 \, d\lambda_3 \, d\lambda_4 \, d\lambda_5 \tag{40}$$

In the pseudoscalar case , W is given by (15) . Inserting (39) in (40) the integrations over λ_4 and λ_5 can be performed to give

$$P(\mu) = \frac{d}{dE} \int e^{\gamma \lambda^2 / k^2} \lambda_1^2 \lambda_2^2 \lambda_3^2 (\lambda_1^2 + \lambda_2^2) \times p_5^2 p_6^2 \, d\lambda_1 \, d\lambda_2 \, d\lambda_3 \, dp_5 \, dp_6 \tag{41}$$

the region of integration being

$$\lambda^2 + \frac{1}{2} (p_5^2 + \frac{1}{4} p_6^2 + p_5 p_6 \mu) + \frac{3}{8} p_6^2 < k^2 \tag{42}$$

where as before

$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 .$$

In (41), taking polar coordinates in the $\lambda_1 \lambda_2 \lambda_3$ space, the angular integrations give a constant factor. The substitution

$$
\begin{aligned}
p_5 &= x \cos\theta \\
p_6 &= x \sin\theta
\end{aligned}
\tag{43}
$$

now leads to

$$
P(\mu) = \frac{d}{dE} \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta \int_0 x^5 \, dx \times \int e^{\gamma \lambda^2/k^2} \lambda^{10} \, d\lambda \tag{44}
$$

the upper limit for being given by

$$
\lambda^2 + \frac{1}{2} x^2 \left(1 + \frac{1}{2}\mu \sin 2\theta\right) < k^2 \tag{45}
$$

The further substitution

$$
x^2 \left(1 + \frac{1}{2}\mu \sin 2\theta\right) = y^2 \tag{46}
$$

now allows us to separate off the integrals over λ and y as constant factors independent of μ, leaving simply

$$P(\mu) = 8 \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cos^2\theta \, d\theta}{\left(1 + \frac{1}{2}\mu \sin 2\theta\right)^3} \tag{47}$$

Although calculated with the pseudoscalar interaction , this expression is independent of the meson coupling and is characteristic of the fact that both α -particles are in S states , or more precisely that the relative coordinates $\underset{\sim}{x}_4$ and $\underset{\sim}{x}_5$ are described by S wave functions.

The integral (47) can easily be evaluated to give

$$P(\mu) = P_A(\mu) = \frac{1 + \mu^2/2}{\left(1 - \mu^2/4\right)^{5/2}} \cos^{-1}\frac{\mu}{2} - \frac{3\mu}{2\left(1 - \frac{\mu^2}{4}\right)^2} \tag{48}$$

This curve , multiplied by $\sin\theta$, is labelled A in Figure 8.

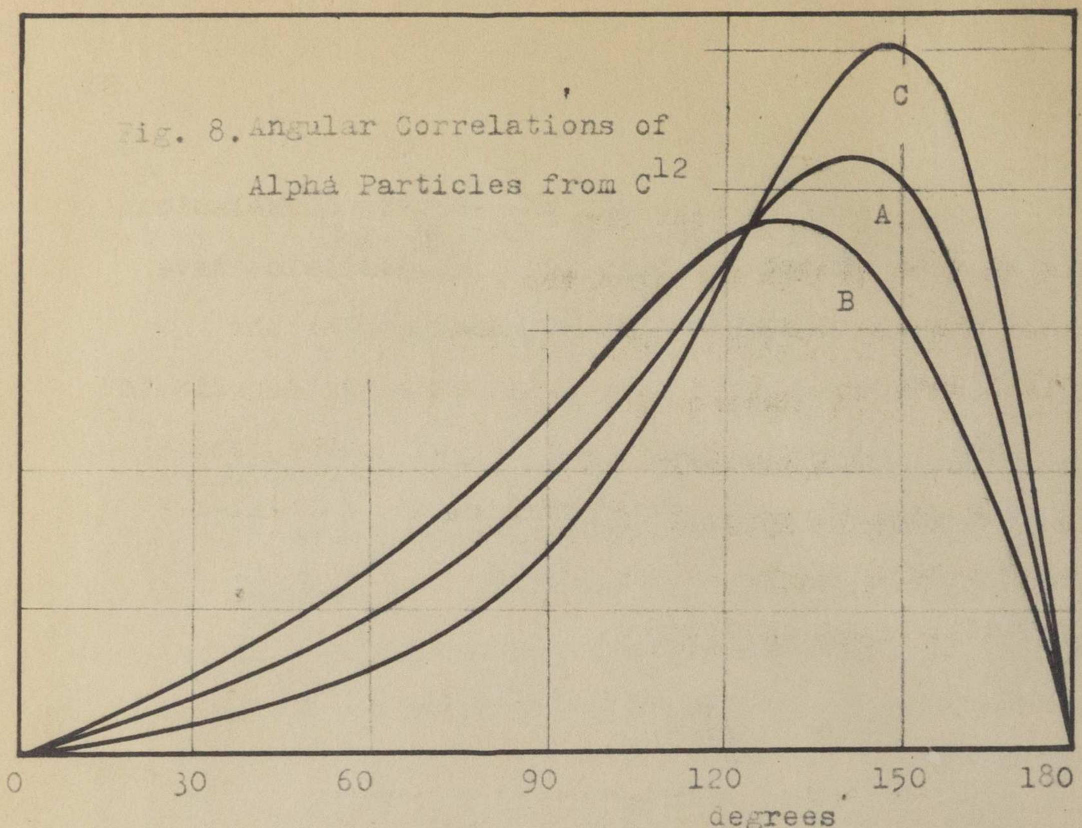


Fig. 8. Angular Correlations of Alpha Particles from C^{12}

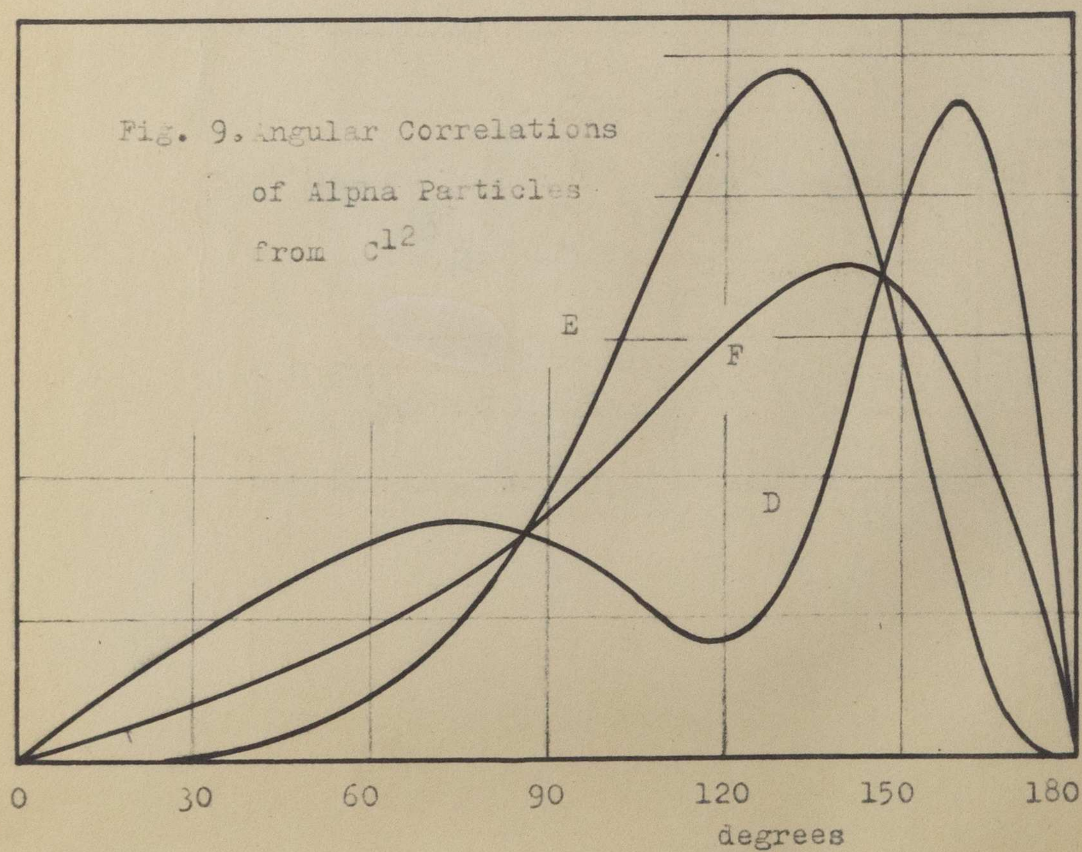


Fig. 9. Angular Correlations of Alpna Particles from C^{12}

Finally we consider the angular correlations arising from states in which the α -particles have higher angular momenta. Let ℓ_1 and ℓ_2 be the orbital quantum numbers associated with the coordinates $\underset{\sim}{x}_4$ and $\underset{\sim}{x}_5$ respectively , and let all other orbital and spin angular momenta be combined and described by the quantum number ℓ_0 . Table 3 shows the particular cases considered.

Table 3

	ℓ_0	ℓ_1	ℓ_2
A	0	0	0
B	1	1	0
C	1	0	1
D	0	1	1
E	1	1	1
F	2	1	1

These are the only cases with both l_1 and l_2 less than 2 . Case A has already been discussed.

These three angular momenta must be combined in such a way that the total angular momentum is zero. In each case there is only one way of doing this. The final state wave function (36) will now contain an additional angular factor Γ under the integral sign , and similarly (37) must be altered by replacing G by $|\Gamma|^2$G .

By way of example , consider case D . Here

$$\Gamma_D = \frac{4\pi}{3} \sum_{m=-1}^{1} (-)^m \, Y_1^m(\underset{\sim}{k}_4) \, Y_1^{-m}(\underset{\sim}{k}_5)$$

$$= \cos\gamma \qquad (49)$$

where γ is the angle between $\underset{\sim}{k}_4$ and $\underset{\sim}{k}_5$.

We now assume that the matrix elements leading to this state are such that

$$\int W \, d\lambda_1 \, d\lambda_2 \, d\lambda_3 = \lambda_4^{2l_1+2} \, \lambda_5^{2l_2+2} \, \mathcal{F}(\lambda_4^2 + \lambda_5^2) \qquad (50)$$

This is true in case A described above and may be regarded as a simple generalization of that case. The dependence on ℓ_1 and ℓ_2 will come from the radial wave functions associated with coordinated $\underset{\sim}{x}_4$ and $\underset{\sim}{x}_5$. In stead of (41) we now have

$$P_D(\mu) = \frac{d}{dE} \int \mathcal{F}(\lambda_4^2 + \lambda_5^2)\, \lambda_4^{2\ell_1} \lambda_5^{2\ell_2} \times \cos^2\gamma \cdot p_5^2 p_6^2 \, dp_5 \, dp_6 \qquad (51)$$

the region of integration being

$$\lambda_4^2 + \lambda_5^2 < k^2,$$

where

$$\lambda_4^2 = \tfrac{1}{2}(\underset{\sim}{p}_5 + \tfrac{1}{2}\underset{\sim}{p}_6)^2$$

$$\lambda_5^2 = \tfrac{3}{8}\, p_6^2$$

$$\ell_1 = \ell_2 = 1$$

and

$$\cos^2\gamma = \frac{(p_6 + 2p_5\mu)^2}{4p_5^2 + p_6^2 + 4p_5p_6\mu} \qquad (52)$$

Taking polar coordinates according to (43) , we find from (51) that

$$P_D(\mu) = \frac{d}{dE} \int_0^{\frac{\pi}{2}} (\sin\theta + 2\mu\cos\theta)^2 \cos^2\theta \, \sin^4\theta \, d\theta$$

$$\times \int_0^{x_0(\theta)} \mathcal{F}\left(x^2 + \frac{x^2}{2}\mu\sin 2\theta\right) x^9 \, dx \qquad (53)$$

where $2k^2 = x_0^2\left(1 + \frac{1}{2}\mu\sin 2\theta\right)$

Just as before , the substitution (46) allows the integral containing $\mathcal{F}$ to be discarded as a constant factor , leaving

$$P_D(\mu) = \int_0^{\frac{\pi}{2}} \frac{(\sin\theta + 2\mu\cos\theta)^2 \cos^2\theta \, \sin^4\theta \, d\theta}{\left(1 + \frac{1}{2}\mu\sin 2\theta\right)^5} \qquad (54)$$

Putting $\cot\theta = t$, we obtain

$$P_D(\mu) = \int_0^{\infty} \frac{(1 + 2\mu t)^2 \, t^2 \, dt}{(1 + \mu t + t^2)^5} \qquad (55)$$

The remaining cases are treated in exactly the same way. The values of $|\Gamma|^2$ required are as follows :

$$|\Gamma_B|^2 = |\Gamma_C|^2 = 1$$

$$|\Gamma_E|^2 = \sin^2\gamma \qquad (56)$$

$$|\Gamma_F|^2 = 1 + \frac{1}{3}\cos^2\gamma$$

It will be noticed that even though $\Gamma_B = \Gamma_C$, these cases will still give different angular correlations because of assumption (50) .

The angular correlation functions $P(\mu)$ obtained are as follows :

$$P_B(\mu) = \frac{16}{5\varpi^7}(5-\mu^2-\mu^4)\cos^{-1}\frac{\mu}{2} - \frac{2}{5\varpi^6}(22-13\mu^2) \qquad (57b)$$

$$P_C(\mu) = \frac{16}{\varpi^7}(1+\mu^2)\cos^{-1}\frac{\mu}{2} - \frac{2\mu}{3\varpi^6}(26+\mu^2) \qquad (57c)$$

$$P_D(\mu) = \frac{2}{\varpi^9}\left(10 - 21\mu^2 + 28\mu^4 + 4\mu^6\right)\cos^{-1}\frac{\mu}{2}$$

$$+ \frac{\mu}{12\varpi^8}\left(188 - 132\mu^2 - 299\mu^4\right) \tag{57d}$$

$$P_E(\mu) = \frac{2}{\varpi^9}\left(1-\mu^2\right)\left(6 + 12\mu^2 + \mu^4\right)\cos^{-1}\frac{\mu}{2}$$

$$- \frac{5\mu}{6\varpi^8}\left(1-\mu^2\right)\left(22 + 5\mu^2\right) \tag{57e}$$

$$P_F(\mu) = 4\,P_E(\mu) + \frac{4}{3}\,P_D(\mu) \tag{57f}$$

In all cases, $\varpi = \sqrt{1 - \frac{\mu^2}{4}}$

The functions P_A, P_B and P_C are shown in Figure 8, P_D, P_E and P_F in Figure 9, all multiplied by $\sin\theta$ and normalized to the same value.

5. 7. Summary and Discussion on Meson Capture by C^{12}

The capture of π^- mesons by the C^{12} nucleus has been considered , making the same assumptions as in the case of capture by He^4 . Since α -particle emission from C^{12} has been observed , a simple α -particle model was used to describe the C^{12} nucleus. It was found that the calculated energy spectrum of the emitted α -particles depended on a single parameter γ , defined by (18) as a function of the parameters a and μ appearing in the He^4 and C^{12} wave functions respectively. Two cases were considered : (a) the α -particles are emitted directly , (b) a Be^8 nucleus in an excited state is emitted , which quickly decays into two α -particles. In the first case the energy spectrum of the α -particles agrees with experiment if we take $\gamma = 12$. The second case requires $\gamma = 0$. Since the experimental spectrum , shown in Figure 5 , is not very accurate , it can be reproduced theoretically by a mixture of cases (a) and (b) in arbitrary proportions provided is given a suitable value ($0 < \gamma < 12$) .

The nuclear radii calculated with these values of the parameters do not agree very well with the known experimental radii , which latter are summarized by Gamow's formula $R = 1.22\ A^{1/3} \times 10^{-13}$ cm . It was found that the radius of He^4 had to be increased , or the radius of C^{12} decreased , from Gamow's values. One expects , of course , that Gamow's formula will overestimate somewhat the radii of these nuclei since they are more stable than neighbouring species. The value required for the C^{12} radius , $\sim 2.4 \times 10^{-13}$ cm , however , seems too big a deviation to be explained in this way. It may be thought that this defect is a consequence of the Gaussian wave functions employed , but calculations , not described here , have also been made with wave functions of the "asymptotic" type (cf. § 4. 3 , equation 26) ; these lead to radii which are in no better agreement with the experimental values than those derived from the Gaussian wave functions.

The explanation of this discrepancy is probably to be found in the fact that the energy spectrum of the emitted α-particles depends mostly on the energy distribution of the α-particles inside the

C^{12} nucleus before the disintegration. This can be seen from the matrix elements , which are in the form of expansion coefficients in the expansion of the initial wave function as a series of free waves. The physical meaning of all this is that the meson is captured by one of the α-particles which immediately blows up , leaving the remaining two α-particles in the state they were in before the explosion. Thus the important thing about the C^{12} wave function is that it should describe a state with a certain energy distribution rather than with a certain radius.

The influence of the meson interaction on the spectrum of the emitted α-particles is negligible , but it can be shown that the energy spectrum of the emitted proton , measured in the frame moving with the centre of gravity of the exploded α-particle , is nearly the same as the spectrum of the protons emitted from free He^4 . We have seen that this is rather insensitive to the meson interaction , and very accurate experiments would be needed to decide in favour of one or other of the meson theories.

Another observable feature of the C^{12} disintegration is the mutual angular correlation of the α-particles , and a wide diversity of these functions has been calculated. As expected , the presence or absence of an intermediate state involving Be^8 has most effect on the calculated angular correlation. The experimental results of M. M. R. ,shown in Figure 7 , indicate that perhaps half the disintegrations involve the emission of Be^8 . The angular correlation functions shown in Figures 8 and 9 show some dependence on the angular momenta of the α-particles , but they all have the same broad characteristics , depending mostly on momentum conservation and on the distribution of mass between the particles. All angular correlations calculated for the case without intermediate state are independent of the bound-state wave functions.

The theory of high energy nuclear events due to Fermi (1950) may be applied to the problems under discussion. This theory is more akin to the evaporation theory than to the perturbation method here employed , since it supposes that energy , here equal to the mass of the π meson , is suddenly liberated in the nuclear volume Ω and that statistical equilibrium is reached before any particles are emitted. The emitted particles will then have energies determined by this statistical equilibrium and given simply by the momentum - space factor $\int dp_E$. This leads to the "standard" energy spectrum in the case of He^4 , in close agreement with the perturbation result , which differs from the "standard" mainly by effects attributed to the exclusion principle. In the case of C^{12} , however , the Fermi theory would give α-particle spectra similar to the case $\gamma = 0$ of Figure 5 , in disagreement with experiment.

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